PROFESSOR: Monday's lecture was all linear equations. And I thought I would start today with nonlinear equations, still first order. And we can't deal with every nonlinear equation. That's too much to ask.

These are going to be made easier by a property called "separable." So these will be separable nonlinear equations. And let me start with a couple of examples and then you'll see the whole idea.

So one example would be the simplest nonlinear equation I can think of, with a y squared. So how to get there? Here's the trick. This is the separable idea. You're going to see it in one shot.

We can separate, put the Ys on one side and the Ds on the other. So I write this as dy over y squared equal dt. I put the dt up and brought the y squared down. So now they're separated, in a kind of hookie way with infinitesimals.

But I'll makes sense out of that by integrating. I'll integrate both sides. I'll integrate time from 0 to t. And I have an initial condition, y of 0, always. And since this one is about y, when t starts at 0, this guy starts at y of 0, up to, this ends at t, so this ends at y of t.

OK. Now the point is, also the problem was nonlinear, we've got two separate ordinary integrals to do. And we can do them.

We can certainly do the right hand side. I get t. And on the left hand side, what do I get? Well, that maybe I better leave a little space to figure out this one.

But the point is we can integrate 1 over y squared. And I guess we get minus 1 over y. So I get minus 1 over y between y of 0 and y of t.

In other words, I'm getting let's see, so what the right, the derivative of the integral of 1 over y squared is minus 1 over y because I always check the derivative gives me that back. So now I'm ready to plug-in those limits.

So I'll do the bottom limit first because it comes with a minus sign, canceling that minus, 1 over y of 0 minus 1 over y of t. Got it. And that equals the other integral, which is just t. So that's the answer as it comes directly from integration.

And we can do more. You can see that finding the solution when these things are separable has boiled down to two integrals. And we could have a function of t here, too.

And that would be allowed, a function of t multiplying this guy, because then I would leave the function of t on that side. And I would have to integrate that. And I would bring the y. You see, I've just separated the y.

In general, these equations look like dy, dt is some function of t divided by some function of y. Maybe the book calls the top one g, I think, and the bottom one f.

And everybody in this room sees that I can put the f of y up there. I can put the dt up there. And I've separated it. OK. So that's sort of the general situation.

This is a kind of nice example, nice example, dy, dt equals y squared. Can we just play with this a little bit? Let me take y of 0 to be 1, just to make the numbers easy.

So if y of 0 is 1, then I have, I'll just keep going a little bit. You do have to keep going a little bit because when you finish the integral right there, you haven't got y equal. You've got some equation that involves y, but you have to solve for y.

So I have to solve that equation for y. Let me just do it. So how would I solve it? And let me take y of 0 to be 1. So now, if I just write it below, I'm at 1 minus 1 over y equals t. Good?

So I'm going to put the 1 over y of t on that side and the t on that side. So if I just continue here, I've got 1 over y of t on this side and, do I have 1 minus t on that side? Yeah. Looking good.

So solution starting from y of 0 equal 1 is y of t equal 1 over 1 minus t. You could do that. You could do that. And I can always, like, mentally I check the algebra at t equals 0. That gives me the answer, 1.

But let's step back and look at that answer. I mean, that's part of differential equations is to do some algebra, if possible, and get to a formula. But if we don't think about the formula, we haven't learned anything. Right there, yes. Good.

So what happens? I want to compare with the linear case that was like e to the t. This was y prime equal y, right? And that led to e to the t. Y prime equals y squared leads to that one.

So first observation. I haven't got exponentials anymore in that solution. Exponentials are just like perfection for linear equations.

For nonlinear equations, we get other functions. Professor Fry had a hyperbolic tangent function in his first lecture. Other things happen.

OK. Now, how do those compare if I graph those? It's just like, why not try? So they both started at 1. And e to the t went up exponentially. E to the t.

And, I don't know, we use exponential. In our minds, we think, that's pretty fast growth. ! mean, that's the common expression, grew exponentially.

But here, this guy is going to grow faster because y is going to be bigger than 1. So y squared is going to be bigger than y. That one's going to grow faster. Faster than exponential.

This has the exponential growth. Pretty fast. Polynomial, of course, some parabola or something would be hanging way down here, left behind in the dust.

But this 1 over 1 minus t, that's going to grow really fast. And what's more, it's going to go to infinity. So that y prime equal y squared, the solution to that doesn't just-- e to the t goes to infinity at time infinity.

At any finite time, we get an answer. Eventually, at t equal infinity, it's gone above every bound. But this one, 1 over 1 minus t is what I want to graph now. I believe that that takes off and at a certain point, capital T, it's going to infinity.

It's blown up. So it's blow up in finite time. Blow up in finite time. And what is that time? What's the time at which the y prime equal y squared has taken off, gone off the charts? T equal--

AUDIENCE: 1.

PROFESSOR: --1. Because when t reaches 1, I have 1 over 0, and I'm dividing by 0, and so that's the blow up. Finite time blow up. OK.

So this can happen for some nonlinear equations. It wouldn't happen for a linear equation. For a linear equation, exponentials are in control. OK.

So that's one nice example. Oh, another nice thing about that example. Well, I say nice if you're OK with infinite series. I just want to compare.

The book mentions the infinite series for these guys because that's an old way to solve differential equations is term-by-term in an infinite series.

It's sort of fun to see the two series. Well, because they're the two most important series in math. Actually, they're the two series that everybody should know. The power series, Taylor series-- whatever word you want to give it for those two guys. So let me do them.

E to the t, I'll put that one first, and 1 over 1 minus t. These are the great series of math. Shall I just write them down and sort of talk through them? Because this is not a lecture on infinite series by any means.

But having these two in front of us, coming from these two beautiful equations, y prime equal y squared and y prime equal y, I can't resist seeing what they look like this way.

So e to the t, do you remember e to the t? It starts at 1. What's the slope of e to the t? At t equals 0. So I'm doing everything-- this series is going to be, both of the series are going to be, around t equals 0. That's my, like, starting point.

So this e to the t thing has a tangent. It has a slope there. And what's the slope of e to the t at t equals 0?

AUDIENCE: 1.

PROFESSOR: 1. It's derivative. The derivative of e to the t is e to the t. The slope is 1. So that tangent line has coefficient 1. That's how it starts. That's the linear approximation. That's the heart of calculus, is this.

But we're going to go better. We're going to get the next term. So what's the next term? That gave us the tangent line.

Now I'm going to move to the tangent parabola. So the parabola has got another is still going to be below the real thing. Can I squeeze in the words "line" and "parab," for "parabola?"

Parabola has bending. I'm really explaining the Taylor series in what I hope is a sensible way. Here is the starting point.

This has the slope. The next term has the bending. The bending comes from what derivative? What derivative tells us about bending? Second derivative. Second derivative tells us how much it curves.

Well, the second derivative of e to the t is still e to the t. So the bending is 1. The bending is also 1. Now that comes in with a factor of a 1/2. There is the tangent parabola. And you will see what these numbers become.

Let me just go to, the third derivative would be responsible for the t cubed term. And its coefficient would be 1 over 3 factorial. So 2 is the same as 2 factorial. 3 factorial is 3 times 2 times 16.

So the numbers here go 1, 6, 24, 120, whatever the next one is. 720 or something. They grow fast. So that series always gives a finite answer. It does grow with t. But it doesn't spike with t.

Now compare that famous series. And of course, this is 1 over 1 factorial, everything consistent. Compare that with the series for 1 over 1 minus t. That's the other famous series that they learned in algebra.

I'll just write it. That's 1 plus t plus t squared plus t cubed plus 1 so on, with coefficient 1. This had 1 over n factorials. Those make the series converge. These don't have the n factorials. This is 1, 1, 1, 1.

And, well, I could check that formula. But do you remember the name for that series? 1 plus t plus t squared plus t cubed plus so on? Algebra is taught differently in many high schools now. And maybe that never got a name.

I guess I would call it the Geometric series. Geometric series. And you see, it's beautiful. It's the other important series. But it's quite different from this one because, what's the difference about this series? Yeah?

AUDIENCE: It goes to infinity.

PROFESSOR: It's a--

AUDIENCE: It goes to infinity.

PROFESSOR: It goes to infinity. But where? At what time? At what value of t is this sum going to fall apart? Blow up? At t equal 1. When have 1 plus 1 plus 1 plus 1, I'm getting infinity.

So this blows up. And of course, we see that it should because this blows up. Left side blows up at t equal 1, the right side blows up a t equal 1.

Where the exponential series, which is the heart of ordinary differential equations, never blows up because of these big numbers in the denominator.

OK, I'm good for this first simple example, y prime equals y squared. It has so much in it, it's worth thinking about.

I'm ready, you OK for a second example? A second important separable equation. I'm going to pick one.

So I'm going to pick an equation that starts out with our familiar linear growth. This could be, you know, last time it was growth of money in a bank. It could be growth of population.

The number of, to a sum first approximation, the rate of growth of the population comes from, like, births minus deaths. And with modern medicine, births are a larger number than deaths. So a is positive, and that grows.

But if we're talking about the, I mean, the United Nations tries to predict, everybody tries to predict, population of the world in future years. And so this could be called the Population Equation. But just to leave it as pure exponential is obviously wrong.

The world can't grow forever. The population can't grow forever. And the, I guess I hope it doesn't grow like 1 over 1 minus t. So this is at least a little slower.

But somehow competition for space, competition for food, for oil, for water-- which is going to be the big one-- is in here. Competition here, of people versus people, a reasonable term, a first approximation, is a y squared, with a minus, is a y squared and with some coefficient.

That's a very famous equation. A first model of population is it grows. But this is a competition term, y against y.

And so, the same would be true if we were talking about epidemics. That's a big subject with ordinary differential equations, epidemiology. Or say, flu. How does flu spread? And how does it get cured?

So partly, people are getting over the flu. But then y against y is telling us how many infected, how many new infections. So we would like to solve that equation. And it's separable.

I can do what I did before, dy over ay minus by squared equal dt. And I can integrate, starting from year of 0. Well, why don't we start from year 2014, with the population y at now-- the present population?

That would be a model that the UN would consider using. That other people with very important interest in measuring population and measuring every resource would need equations like this.

And then they would put on more terms, like a term for immigration. All sorts, many improvements have to go into this equation.

Let me just look at this as it is. Well, I've got two choices here. Well, it's this integral that I'm looking at. That is a doable integral. It's the type of integral that we saw in the Rocket problem.

The Rocket problem was more constant minus. This was a drag term, when we were looking at rockets. And this was a constant, say, gravity. So it was still a second degree. Still second degree, but a little different.

This has the linear in second degree terms. If you look up that integral, you'll find it. Or there's a systematic way to do it. That's in 1801, I guess, called partial fractions. It's not a lot of fun.

I don't plan to do it. It's in the book. Has to be because that's the way you can integr-- you can integrate polynomials over polynomials by partial fractions.

That's what they're for, but there's a neat way to do this one. There's a neat trick that Bernoulli discovered to solve that equation, to turn it into a linear equation. And of course, if we can turn it into a linear equation, we're on our way.

So the neat trick is let z be 1 over y. You can put this in the category of lucky accidents, if you like. So now I want an equation for z. So I know that dz, dt if I take the derivative of that, that's y to the minus 1. So it's minus 1 y to the minus 2 dy, dt.

That's the chain rule. Take the derivative of 1 over 1, you get minus 1 over y squared. Multiply by the derivative of what's inside. That's the chain rule.

OK. So I plan to substitute those in here. So dy, dt, let's see. Can you see me? You can probably do it better than me.

So dy, dt is minus. I'll bring that up. Dy, dt I'm going to put-- I hope this'll work all right-- for dy, dt, I'm going to put in dz. Using this, I'm going to put minus y squared dz, dt. Did that look right? I don't think I'm necessarily doing this the most brilliant way.

But dy, dt-- I put this up here and I got that-- equals ay. So that's a over z. Oh, y Is 1 over z. So get this, I want all Zs now. So that's this part. And ay is over z minus by squared is minus b over z squared.

Would you say OK to that? I've got Zs now, instead of Ys. I just took every term and replaced y by 1 over z. Y is 1 over z and dy, dt I can get that way. OK. Yeah.

Now what? Now look what happens, if I multiply through by z squared or by minus z squared. Let me multiply through by minus z squared. I get dz, dt.

Multiplying by minus z squared gives me a minus az. And what do I get when I multiply this one by minus z squared?

AUDIENCE: Plus b.

PROFESSOR: I get plus b. Look what happened. By this, like, some magic trick. You could say, all right. That was just a one time shot.

But it was a good one. We ended up with a linear equation for z. A linear equation for z. And we solved that equation last time.

So let me squeeze in the solution for z, and then elsewhere. So what was the solution for z of t? It was some multiple of, no, yeah. This is perfect review of last time.

We have a constant times z. And so that's going to go into the exponential. This will be the, it's a minus a, notice. That will be the, what part of the solution is that one called?

That's the null solution. The null solution, when b is o. And now I add in a particular solution. A particular solution.

And one good particular solution is choose the z to be a constant. Then that'll be 0. So I want that to be 0. So what constant z makes that 0?

I think it's b over a, don't you? I think b over a. Does that work good? That's every null solution plus one particular solution.

Let me say now, and I'll say again, looking for solutions which are steady states, b over a-- of this particular solution, that particular solution made this 0.

So it made this 0. So it's a solution that's not going anywhere. It's a constant solution. It's a solution that can live for all time. OK. B over a. Let me put that word there, steady state. OK.

And now I would want to match the initial conditions using c. Yeah. I'd better do that. OK. And I have to get back to y. I have y is 1 over z.

So I'm going to have to flip this upside down. I'm going to have to flip this upside down is what will actually happen. Let me make it easy to flip.

Let me, I'll change c, which is just some constant to some constant d over a. So then it's a is everywhere down below. And I just write it here in the middle. That makes it easier to flip. So finally I get their solution.

Solution to the population equation. But that's the famous word for it, the Logistic equation. This is section 1.7 of the text on the differential equations in linear algebra. It's a very, very much studied example. It's a great example.

It fits the growth of human population with some, it's our first level approximation to growth of or other populations or other things. It's a linear term giving us exponential growth, and a quadratic term of competition slowing it down. And let's see that slow down.

So now that was a bit of algebra. Much nicer than partial fractions. The bit of algebra just came from this idea of going to z. And now I want to go back to y.

So y is 1 over z. So it's a over d e to the minus at plus b. That's our solution. A and b came out of the equation. And d is going to be the number that makes the initial value correct.

So at t equals 0, I would have y of 0, whatever the initial population is, is a over d. T Is 0, so that's just 1 plus b. So that tells me what d is. D equals something.

It comes from y of 0. So the answer, let me circle that answer. That answer has three numbers in it, a, b, and d. a and b come from the equation. D also involves the initial starting thing, which is exactly what it showed.

So you could say we've solved it. But if you ever solve an equation like this, you want to graph it. You want to graph it. So let me draw its graph. This is important picture.

So here is time. Here is population. Here's, maybe it started there. This is times 0. And now I want to graph this. I want to graph that function.

Really, this is where we're going somewhere. What happens for a long time? At t equal infinity, what happens to the population?

Does it grow, like e to the t? Just remember the examples here. We had a growth like e to the t. We had a growth faster than e to the t that actually blew up.

What about this guy? What will happen as t goes to infinity with that population? It goes to?

AUDIENCE: A over b.

PROFESSOR: A over b. A over b. That's the key number in the whole thing. It keeps growing, but it never passes a over b. This is y at infinity. That's the final population.

So how does it do this? If I draw this graph-- and what about negative time? Let's go backwards in time. What is it at t equals minus infinity? Then you really see the whole curve.

At t equal minus infinity, what is this doing?

AUDIENCE: 0 infinity.

PROFESSOR: It's 0. Good. Good. Good. T equal minus infinity, this is enormous. This is blowing up. It's in the denominator. We're dividing by it. So the whole thing is going to 0.

So here's what the logistic curve looks like. It creeps up. And it's beautifully, there's a point of symmetry here. The growth is increasing here.

And then, as a point of inflection you could say, growth is bending upwards for a while. At this point, it starts bending downwards. From that point on, ooh, let's see if I can draw it. It'll get closer, and exponentially close.

That wasn't a bad picture. The population here is half way. Here, the population, the final population, is a over b. And just by beautiful symmetry, the population here is a 1/2 of a over b. At this point.

If this was the actual population of the world we live in-- I think we're pretty close to this point. I believe, well, of course, nobody knows the numbers, unfortunately, because the model isn't perfect. If the model was perfect, then we could just takes the census and we would know a and b.

But the model isn't that great. But it's sort of, we're at a very interesting time, close to a very interesting time. I believe that with reasonable numbers, this a over b might be maybe 12 billion.

And we might be, I think we're a little above six billion. I think so. So we're a little bit past it. This is now. This is halfway. That's the halfway point. It's perfectly symmetric. It's called an S curve.

And many, many equations in math biology involve S curves. So math biology often gives rise, with simple models, to a kind of problem we've had here with a quadratic term slowing things down. Enzymes, all kinds of.

Ordinary differential equations are core ideas in a lot of topics, lot of areas of science. OK. Do I want to say more about the logistic equation?

I guess I do want to distinguish one thing. Yeah. One thing about logistic equations and will of course come back to this. OK. Let me look at that logistic equation.

Here's my equation. So I've managed to solve it. Fine. Great. Even graph it. But let me come back to the question, suppose I just look at.

I can see two constant solutions, two steady states, two solutions where the derivative is 0. So nothing will happen. So in other words, I want to set this thing set to 0 equal to 0 to find steady solutions.

Steady means the derivative is 0. So this side has to be 0. So what are the two possible steady states where, if y of 0 is there, it'll stay there?

AUDIENCE: 0.

PROFESSOR: 0. Y equals 0 is one. And the other?

AUDIENCE: A over b.

PROFESSOR: Is a over b. So steady equal to 0. And I get two steady states. Let me call them capital Y equals 0 because that's certainly 0 of, if we have 0 population, we'll never move. Or setting this to 0, ay is by squared cancel y's divide by b a over b.

So the two steady states are here. That's a steady state and that's a steady state.

Those are the only two in this problem.

You see how easy that was to find the steady states? That's an important thing to do. And then the other important thing to do is to decide, are those steady states stable? When the population's near a steady state, does it approach that, does it go toward that steady state or away?

So what's the answer? For this steady state, that steady state, y is a over b. Is that stable or unstable? So I'll write the word stable. And I'm prepared to put in "un," unstable, if you want me to.

This is a key, key idea. And with nonlinear equations, you can answer this stability stuff without formulas. Without formulas. That's the nice thing. And then that comes in a later class. But here's a perfect example.

So do we approach this answer or do we leave it? We approach it, the solutions. This is stable, yes. And here's the other stationary point, capital Y. The other steady state is that nothing happens.

So now if I'm close to that, if y is a little number, like 2, will that 2 drop to 0, will it approach this steady state, or will it leave it?

AUDIENCE: Leave it.

PROFESSOR: Leave it. So this steady state is.

AUDIENCE: Unstable.

PROFESSOR: Unstable. Unstable. Right. Right. With linear equations, we really only had one steady state, like 0. Once it started, it took off forever. Here, it doesn't go infinitely high. It bends down again to that limit, that carrying capacity is what it's called, a over b.

I guess I hope you think a nonlinear equation like got a little more to it. Little more interesting, but a little more complicated, than linear equations. Yep. Yep. Yep.

And similarly, the rocket equation, we could at the right time soon in the course, ask the same thing, a rocket equation was something like that. What are the steady states? Are they stable? Are they unstable? Can you find a formula?

Here. This. We got a formula. And there are other nonlinear equations, which we'll see. OK. I could create more separable equations, but I guess I hope that you see with separable equations, you just separate them and integrate a y integral and a t integral.

Is that OK any question on this nonlinear separable stuff? Differential equations courses and the subject tends to be types of equations as can solve.

And then there are a hell of a lot of equations that are not on anybody's list, where you could maybe solve them by an infinite series, but not by functions that we know.

OK. I'm ready to do the other topic for today. It's the topic that I left incomplete on Monday. So I'm staying with first order equations, but actually this topic is essential for second order equations. So I'm going to topic two for today.

So topic two will involve complex numbers. So we have to deal with complex numbers. And the purpose of introducing these complex numbers is to deal with what we met last time when the right hand side, the forcing term, was a cosine.

Typical alternating current, oscillating, rotating, rotation. All these things produce trig functions. Maybe rotation is more of a mechanical engineering phenomenon. Alternating current more of an EE phenomenon. But they're always there.

And what was the point? The point was we had some linear equation, and we had some forcing by something like cos omega t. Or it could be A cos omega t and B sine omega t.

Either just cosine alone, or maybe these come together. And then the solution was y equals some combination of those same guys. In other words, what I'm saying is cosines are nice right hand forcing functions. Fortunately, because we see them all the time.

But they do lead to cosines and sines. I emphasized that last time. If we just have cosines in the forcing function, we can't expect that there's any damping, we can't expect only cosines. We have to expect some sines.

In other words, we have to deal with combinations of them. And the question is, how do you understand cos omega t plus 3. Or let me take a first example.

Example-- cos t plus sine t. That's a perfect example. So what is omega here in this example that I'm starting with?

AUDIENCE: 1.

PROFESSOR: 1. So I just read off the coefficient of t is 1, 1 hertz here. But we have got this combination. And the question is, how do we understand that cosine plus sine? Two very simple functions, but they're added, unfortunately.

And there's a much better way to write this so you really see it. You really see this. That's called a sinusoid. And the rule that want to focus on now is that everything of that kind, of this kind, of this kind, of a cosine plus a sine, can be compressed into one term. One term.

Of course, it's got to have two constants to choose because that had an a and a b. This had an m and an n. This had a 1 and a 1. But the term I'm looking for is some number R times a pure cosine of omega t, but with a phase shift.

So you see there are two numbers here to choose. It's really like going from rectangular to polar. Say in complex numbers, let's just remember the first fact about a complex number. If the real part is 3, and the imaginary part is, let's say 2, then here's a complex number, 3 plus 2 i.

So this was the real axis. This was the imaginary axis. I went along 3, I went up 2, I got to that number. There it is. I plotted the number 3 plus 2 i in the complex plane.

And for me, that number 3 plus and so on, really saying something important. And maybe it's not entirely new. I'm saying something important about complex numbers, this is their rectangular form. Something plus something.

That form is nice to add to another complex number. If I added 3 plus 2 i to 1 plus i, what would I get?

AUDIENCE: 4 plus 3 i.

PROFESSOR: 4 plus 3 i. But if I multiply, multiply, 3 plus 2 i times, let's say I square it. I multiply 3 plus 2 i by 3 plus 2 i. What do I get?

If I do it with this rectangular form, I get a mess. I can't see what's happening. It's the same over here. This is like having a 1 and a 1, with an addition. This is like a polar form where it's one term.

OK. So let me answer the question here and then let me answer the question there. And then you've got a good shot at what complex numbers can do, and why we like the polar form for squaring, for multiplying, for dividing.

What's the polar form? Well, I'm using that word "polar" in the same way we use polar coordinates. What are the polar coordinates of this point?

They're the radial distance, which is what? So what's that distance? That's the R you could say. It corresponds to this R here. So I'm just using Pythagoras. That hypotenuse is what?

AUDIENCE: Square root of 13.

PROFESSOR: Square root of 13. Thanks. 9 plus 4, square root of 13. And what's the other number that's locating this in polar coordinates? The angle.

And the angle. What can we say about that angle? Let's call it phi is-- what's the angle? Well, it's some number. It's between 0 and pi over 2, I'm sure of that. What do I know about that angle?

I know that this is 2 and this is 3. So that's telling me the angle. Well, what is that really telling me immediately? It's telling me the.

AUDIENCE: Tangent.

PROFESSOR: Tangent of the angle. So the tangent of the angle is 2 over 3. And the magnitude is square root of 13. OK. So those beautiful numbers, 2 and 3, have become a little weirder.

Square root of 13, inverse tangent of 2/3. You could say, well, that's not so nice. What was I going to do? I was going to try squaring that number.

So if I square 3 plus 2 i, or if I take the 10th power of 3 plus 2 i, or the exponential, all these things, then I'm happy with polar coordinates.

Like, what would be the magnitude of the square? And where will the square of that number, so I want to put in 3 plus 2 i squared, which I can figure out in rectangular, of course-- a 9, and 6 i, or 12 i, or 4 i squared, stuff like that. It's not pleasant.

What's the magnitude, what's the R for this guy? What's the size of that number squared? Yes? Say that again.

AUDIENCE: 13.

PROFESSOR: 13. Right. I just have to square this square root so I get 13. And the angle will be, what's the angle for the square there? I don't want a number. I guess I'm just doing this. R e to the i phi squared is R squared. And what's the angle here?

E to the i phi squared is e to the 2 i phi. It's the angle doubled. E to the 2 i phi. The angle just went from phi to 2 phi. The lengths went from square root of 13 to 13.

Squaring, multiplying is nice with complex numbers. Maybe can I before I go on and on about complex numbers, I should ask you, how many know all this already? Complex numbers are familiar? Mostly. Correctly, with a wiggle. OK.

I won't go more about complex numbers. Let me come back to my question here. Let me come back to the application. So here it is with complex numbers. Here it is with sinusoids.

And the little beautiful bit of math is that the sinusoid question goes completely parallel to the complex number question. So you have an idea on those complex numbers. We'll see them again. Let me go to this.

So I want this to be the same as this, OK. Maybe I'm going to have to use a new board for this. Can I start a new board?

So I want cos t plus sine t to be some number R times cosine of t 1. I can see omega's 1, so I just put to 1 minus some angle. OK. And I want to choose R and phi to make that right.

You see what I like about it? This tells me the magnitude of the oscillation. It tells me how loud the station is. When I see cos t and, separately, sine t, or I might see 3 cos t and 2 sine t.

3 cos t is a cosine curve. 2 sine t is a sine curve shifted by 90. I put them together, it bumps, it bumps, bumps. Not completely clear.

It seems to me just beautiful that if I put together a cosine curve that we know, that starts at 0, with a sine curve that starts at 0, the combination is a cosine curve. Isn't that nice?

I mean, you know, that sometimes math gets worse and worse whatever you do. But this is really nice that we can put the two into one. But you see, it's going to-- well, let's do it.

What would R and phi be here? So I'll use a trig fact here. A cosine of a difference of angles, so this is R times cosine t, do you member this? This was the whole point of going to high school. Plus sine t sine phi.

So now, how do I get R and phi? I use the same idea that worked last time. I match the cosine terms and I match the sine terms. So the cosine t has a 1. 1 cosine t is R cos phi. That's what's multiplying cosine t. And the sine t has a 1. And that has to agree with R sine phi.

So I'm in business if I solve those two equations. And well, they're not linear equations. But I can solve them. Of course, the one fact that you never forget is that sine squared plus cosine squared is 1. Right?

So if I square that one, and square that one, and add, what will I get? 1 squared and 1 squared will be 2, on the left hand side.

On the right hand side, I'll have R squared cos squared, R squared cos squared, and plus R squared sine squared. And what's that? What's R squared cosine squared plus r squared sine squared?

AUDIENCE: R squared.

PROFESSOR: It's just R squared. So all that added up to R squared. In other words, it's just like polar coordinates. R is the square root of 2. That's telling us the magnitude of the response. Square root of 2.

You see, it's just like complex numbers. It's like the cosine gave us a real part and the sine gave us an imaginary part. And R was the hypotenuse. And that's really nice.

So R is the square root of 2. OK. Now, the angle is never quite as nice. But how can we get something about an angle out of there?

All we could get in this case here was the tangent of the angle. And I'll be happy with that again here because it's a totally parallel question. How am I going to get the tangent of the angle?

What do I have? From these two equations, I want to eliminate R. So how do I eliminate R? What do I do? Divide. Divide.

I guess if I want tangent sine over cosine, I'll divide this one in the top by this one in the bottom. So I take the ratio. That'll cancel the Rs perfectly. It'll leave me with 10 phi. And here it happens to be 1. OK.

So what have I learned? I've learned that when these two add up together, they equal what? R square root of 2. You see how easy it is.

Square root of 2 came from the square root of 1 plus. It's like Pythagoras. Pythagoras going in circles, really. Times the cosine of t minus. And what is phi? Its tangent is 1, so what's the angle phi?

AUDIENCE: Pi over 4.

PROFESSOR: Pi over 4. Right. So that's the sinusoidal identity when the numbers are 1 and 1. But you saw the general rule. Let me just take it.

Suppose this is the output, and cos omega t plus n sine omega t. What is the gain? What's the magnitude, the amplitude, the loudness of the volume in this when I'm tuning the radio?

What's the R for this guy? What's this R? If we just follow the same idea. So if we have m times a cosine and n times a sine, what's your guess? What's your guess for R, the magnitude?

I'm guessing a square root of what? Yeah? You got. What is it? n squared--

[INTERPOSING VOICES]

PROFESSOR: Plus n squared. Way to go. M squared plus N squared. And the angle is like the phase shift. I'm not great at graphing, but let me try to go back to my simple example.

If I tried to add up on the same graph cosine t, which would start from 1 and drop to 0, go like that, right? Something like that would be cosine. And now I want to add sine t to that. So that climbs up to 1 back to 0, down.

And now if I add those two, this formula is telling me that it comes out neat. Neatly. That one plus that one is another sinusoid with height square root of 2. If I had different chalk, I've got at least a little bit different.

But does it start here? Of course not. It starts here, I guess. But it goes up, right? Because this comes down, but this is going up. All together, it's up to, where is the peak? Where is the peak on the sum? So I'm adding, everybody sees what I'm doing?

I'm adding a cosine curve and a sine curve. And it goes up. And where does it peak? What angle is it going to peek at? What's the biggest value this gets to?

AUDIENCE: [INAUDIBLE].

PROFESSOR: At pi over 4, it'll peak. At pi over 4, it'll be the cosine of 0, which is 1. It's height'll be the magnitude, the gain, square root of 2.

So it'll peak at pi over 4, which is probably about there, right? Peak at pi over 4 and, I don't know if I got it right frankly. I did my best. That's the sum. That right there.

The first key point is it's a perfect cosine. The second key point is it's a shifted cosine. The third key point is its magnitude is the square root of 1 squared plus 1 squared, or n squared plus n squared, or a squared plus b squared.

So that's the sinusoidal identity. A key identity and being able to deal with forcing terms, source terms, that are sinusoids. OK.

Now, I'm going to take one more step since we have just like 10 minutes left, and let the number i get in here properly. Get a complex number to show up here. OK.

Before I start on this, let me recap. Let me recap today's lecture. It started with nonlinear separable equations. And a great example was the logistic equation up there, with the S curve. That took half the lecture.

The second half of the lecture has started with things real with sinusoids that are combinations of cosine and sine and has written them in a one term way.

And now I want to get the same one term picture from using complex numbers. OK. OK. And everything I do would be based on this great fact from Euler that e to the i omega t. The real part is cosine omega t. And the imaginary part is sine omega t.

That's a central formula. Let me draw it rather than proving it. Let me draw what that means. I'm in the complex plane again. Real part is the cosine. The imaginary part is the sine.

That number there is e to the i omega t because it's got that real part and that imaginary part. And what's its magnitude?

What's the R, the polar distance for cos omega t plus, for this number, which is for this number? What's the hypotenuse here? Everybody knows.

AUDIENCE: 1.

PROFESSOR: 1. Hypotenuse is 1. Cos squared plus sine squared is 1. So e to the i omega t is on a circle of radius 1. That's the most important circle in the complex world, the circle of radius 1.

And all these points are on it. And their angles are omega t. And as t increases, the angle increases, and you go around the circle. You've seen it. Physics couldn't live without this model. OK.

So that's basically what we have to know. And now, how do we use it? Well, the idea is to deal with the equation. Like, the equation I had last time was dy, dt equals y plus cos t.

That gave us some trouble because the solution didn't just involve cosines, it also involved sines. Yeah. So I want to write that equation differently, in complex form. And this is the key point here.

So I'm going to look at the equation dz, dt equals z plus e to the i t. Well, I'll make that cos omega t just to have a little more, the units are better, everything's better if I have a frequency there. Units of this are seconds and the units of this are 1 over seconds.

Now, question. What's the relation between the solution z to that complex equation and the solution y to that equation? Of course, they have to be related, otherwise it was stupid to move to this complex one.

My claim is that complex equation is easy to solve. And it gives us the answer to the real equation. And what's the connection between y and z?

AUDIENCE: So y's the real part.

PROFESSOR: Y is, exactly, say it again.

AUDIENCE: The real part--

PROFESSOR: Of z. Y is the real part of z. So that gives us an idea. Solve this equation and take it's real part. If I can solve this equation without getting into cosine and sine separately and matching, I can stay real. I solve the equation by totally real methods up to now.

Now I'm going to say, here's another approach. Look at the complex equation, solve it, and take the real part. You may prefer one method. You may like to stay real. In a way, it's a little more straightforward.

But the complex one is the one that will show us, it brings out this R, it brings out the gain, it brings out the important-- engineering quantities are important, if I do it this way. Now, I believe that the solution to that is easy.

Actually, it is included in what I did last time. It's a linear equation with a forcing term that's a pure exponential. And what kind of solution do I look for? I'm looking for a particular solution.

If I see an exponential forcing term, I say, great. The solution will be an exponential. So the solution will be sum. Z is sum capital Z e to the i omega t.

Plug it in. What happens if I plug that in to find capital Z, which is just a number? Right. This is my method. This is a linear equation, with one of those cool right hand sides, where the solution has the same form with a constant, and I just have to find that constant.

So I plug it in. Dz, dt. Take the derivative of this, z i omega will come down. E to the i omega t. Z is just this, z to the i omega t. And this is just 1 e to the i omega t.

So I plugged it in, hoping things would be good. And they are because I can cancel e to the i omega t, that's the beauty of exponentials, leaving just a 1 there. So what's capital Z? What's capital Z then?

I've got a z here. I better bring it over here. And I've got the 1 there. I think the z is 1 over. When I bring that z over here, do you see what I'm getting? It's all multiplying this e the i omega t. It's a number there.

Z times i omega and comes over as a minus z. What do I have multiplying z here? I see the i omega. And what else have I got multiplying the z?

AUDIENCE: Minus.

PROFESSOR: A negative 1 because it came over with a minus sign. Done. Equation solved. Equation solved. Complex equation solved. So the point is, the complex equation was a cinch. We just assumed the right form, plugged it in, found the number, we're done.

But there's one more step, which is what? Take the real part. So I have to take the real part of this. So the correct answer is y is the real part of that number, 1 over i omega minus 1 times e to the i omega t.

I'm tempted to stop there, but just with a little comment. How am I going to find that real part? And what form will it have? What form will that real part have? Yeah, maybe just to say what form will it have?

The real part, it's going to be a sinusoid. But I have a complex number multiplying this guy. The real part is going to be exactly of the form we-- well, of course, it had to be the form because that was another way to solve the equation.

It's going to be some number. And I'll call it g, for gain, times the real part. And so the real part will be a cosine. Yeah, it's just perfect. A cosine of omega t. And there'll be a phase. Yeah.

i haven't taken that step fully. I got to that fully. And then I said that that, if I use some complex arithmetic, will come out to be this.

And you see the beauty of that answer, which was way better than a sum of sines and cosines. We see the gain. We see the amplitude. And we see the phase shift. Yeah.

So I don't know, that would be a good exercise in complex numbers. Find g and find phi, in taking the real part of this thing. Yeah. It's a pure exercise in using complex numbers. I don't feel like doing it today.

If we do it, you just see a lot of formulas. Here, you see the point. The point was that the complex equation could be solved in one line. We just did it.

But that left us the problem of taking the real part. That was the e to the i omega t there. Left us the problem of taking the real part. And that's a practice with complex arithmetic.

So you've got the choice. Either stay real-- sign plus cosine. And then use the sinusoidal identity, polar form. Or get the polar form from here. Same answer both ways.