Buckling

general

Up to this point, the stress and deflections have been proportional to an applied load:

e.g. $\sigma_x = M \cdot \frac{y}{I}$ bending stress proportional to moment

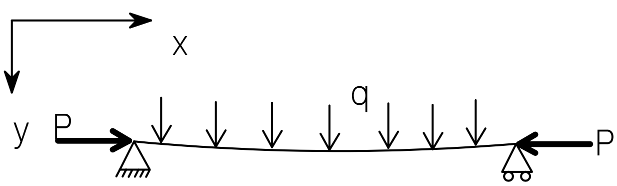
maximum deflection of a simply supported beam subject to uniform load per unit length q =>

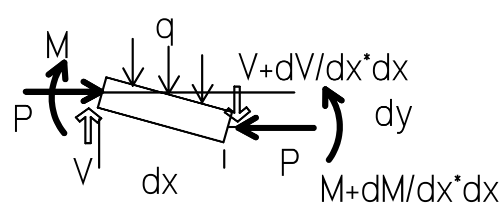
$y_{max} = \frac{5}{384} \cdot \frac{q \cdot L^4}{E \cdot I}$ deflection proportional to the uniform load.

This is not always the case, such as when compressive loads with/without lateral loads act on a column (beam). Moments, stresses and deflections will NOT be proportional to axial loads, but will be dependent (not proportional) to deflections, thus sensitive to slight initial deflections and/or eccentricities in the application of the load.

Euler buckling (derived from general case of beam-columns including lateral load q(x))

consider:





$\Sigma F_{y\_dir} = 0 \Rightarrow \left(V + \frac{d}{dx}V \cdot dx\right) - V + q \cdot dx = 0 \Rightarrow q(x) := -\frac{d}{dx}V(x)$ (1)

and $\Sigma M_A = 0 \Rightarrow M + \left(\frac{d}{dx}M\right) \cdot dx - M - \left[V + \left(\frac{d}{dx}V\right) \cdot dx\right] \cdot dx - q(x) \cdot dx \cdot \frac{dx}{2} - P \cdot \frac{d}{dx}y \cdot dx = 0 \Rightarrow$

$\frac{d}{dx}M - P \cdot \frac{d}{dx}y = V$ (2)

as in previous bending: $M(x) := -E \cdot I \cdot \frac{d^2}{dx^2}y(x) \Rightarrow -\frac{d^2}{dx^2}M(x) = E \cdot I \cdot \frac{d^4}{dx^4}y(x)$

using (1) and (-) the derivative of (2) wrt x; $-\left[\frac{d}{dx}\left(\frac{d}{dx}M - P \cdot \frac{d}{dx}y\right)\right] = -\left(\frac{d}{dx}V\right) = q(x) \Rightarrow$

$E \cdot I \cdot \frac{d^4}{dx^4}\left(y(x) + P \cdot \frac{d^2}{dx^2}y(x)\right) = q(x)$ or setting $k := \sqrt{\frac{P}{E \cdot I}} \Rightarrow$

$\frac{d^4}{dx^4}\left(y(x) + k^2 \cdot \frac{d^2}{dx^2}y(x)\right) = \frac{q(x)}{E \cdot I}$ euler buckling uses q = 0 which we will do now general solution is: $y(x) := A\cdot\sin(k\cdot x) + B\cdot\cos(k\cdot x) + C\cdot x + D$

check => $\frac{d^4}{dx^4}\left(y(x) + k^2\cdot\frac{d^2}{dx^2}y(x)\right) \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

now apply to column with pinned ends:

boundary conditions are: y(0) = y(L) = 0

and $\frac{d^2}{dx^2}y(0) = \frac{d^2}{dx^2}y(L) = 0$ (no bending moment at the ends)

y(0) = 0 => B + D = 0
y(L) = 0 => $A\cdot\sin(k\cdot L) + B\cdot\cos(k\cdot L) + C\cdot L + D = 0$

$\frac{d^2}{dx^2}y(0) = 0$ => $-k^2\cdot A\cdot\sin(k\cdot 0) - k^2\cdot B\cdot\cos(k\cdot 0) = -k^2\cdot B = 0$ => B and D = 0

$\frac{d^2}{dx^2}y(L) = 0$ => $-k^2\cdot A\cdot\sin(k\cdot L) = 0$ => $A\cdot\sin(k\cdot L) = 0$ => C = 0 from the y(L) = 0 relation above

this leaves $A\cdot\sin(k\cdot L) = 0$ which has a non trivial solution only when $\sin(k\cdot L) = 0$ => $k*L = n*\pi$

recall that k^2 = P/(E*I) => k^2 = $\left(\frac{n\cdot\pi}{L}\right)^2$ = P/(E*I) or .... solution defining that force P as P_{cr}

when $P_{cr} := (n\cdot\pi)^2\cdot\frac{E\cdot I}{L^2}$ the displacement is then: $y(x) := A\cdot\sin(k\cdot x)$ where A can be any value. i.e.

with P < Pcr the trival solution applies y = 0, but at Pcr y(x) can be >0 and arbitrary.

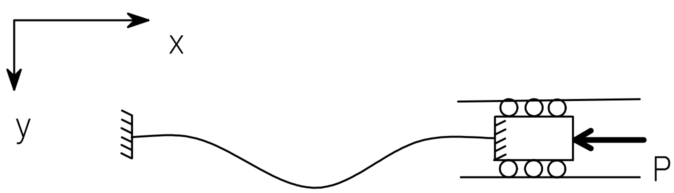
minimum P_{cr} occurs when n = 1 => $P_{cr} := \pi^2\cdot\frac{E\cdot I}{L^2}$ sometimes labeled P_E

let's look at a few other sets of boundary conditions and determine the P_{cr} by inspection:

A. clamped - free

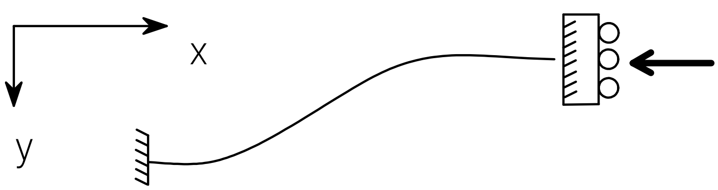
P_{cr}

B. clamped - clamped



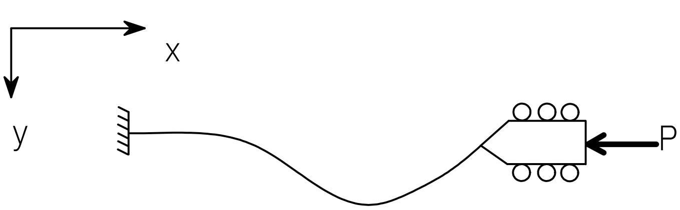
P_{cr}

C. clamped - clamped, free to translate



P_{cr}

D. clamped - pinned, not free to translate



P_{cr}

this one is not obvious (at least to me!!) let's apply boundary conditions to the general solution:

$$y(x) := A \cdot \sin(k \cdot x) + B \cdot \cos(k \cdot x) + C \cdot x + D$$

boundary conditions are: $y(0) = \frac{d}{dx}y(0) = 0$; clamped at 0and $y(L) = \frac{d^2}{dx^2}y(L) = 0$ (no displacement or bending moment at L)

$y(0) = 0 \Rightarrow B + D = 0$ (1)

$\frac{d}{dx}y(0) = 0 \Rightarrow A*k + C = 0$ (2)

$y(L) = 0 \Rightarrow A\cdot\sin(k\cdot L) + B\cdot\cos(k\cdot L) + C\cdot L + D = 0$ (3)

$\frac{d^2}{dx^2}y(L) = 0 \Rightarrow -k^2\cdot(A\cdot\sin(k\cdot L) + B\cdot\cos(k\cdot L)) = 0 \Rightarrow A\cdot\sin(k\cdot L) + B\cdot\cos(k\cdot L) = 0$ (4)

$\Rightarrow C\cdot L + D = 0$ (3) has only trivial solution A = B = C = D = 0

solve for A in terms of B using (1), (2) and (3)

(3) => C = -D/L (1) => D = -B => C = B/L

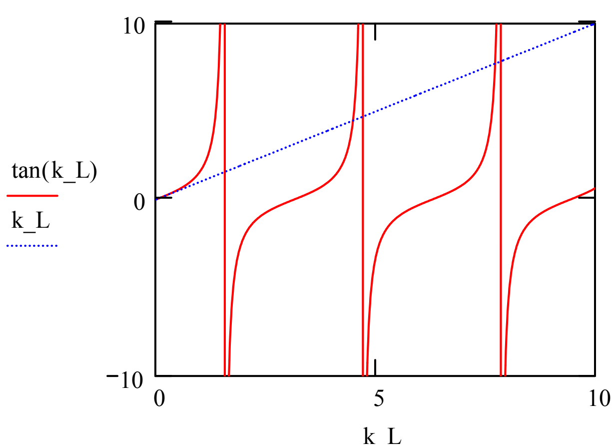
(2) => A = -C/k = -B/(k*l)

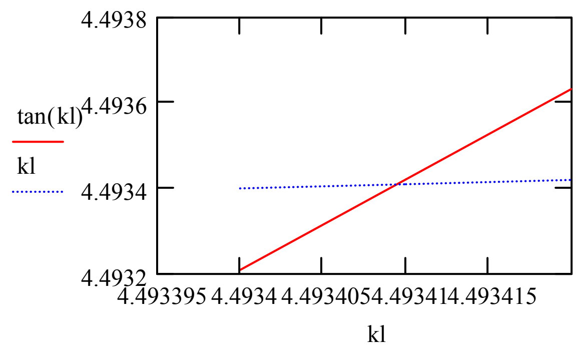
(4) => $B\cdot\left(\frac{-\sin(k\cdot L)}{k\cdot L} + \cos(k\cdot L)\right) = 0 \Rightarrow \frac{-\sin(k\cdot L)}{k\cdot L} + \cos(k\cdot L) = 0 \Rightarrow$

k*L = sin(k*L)/cos(k*L) = tan(k*L) a transcendental equation - solve graphically: $k\_L := 0, 0.01 .. 10$

intersection approaches k*L = n*π/2, with first occurrence at ~ $3\cdot\frac{\pi}{2} = 4.7$ is between 4.49 and 4.5 by trial and error

$kl := 4.4934, 4.49341 .. 4.49342$





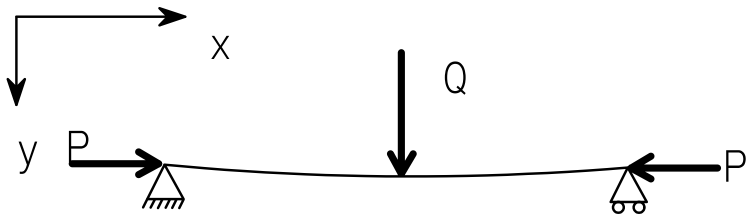
value found by successive iteration $k\_L := 4.49341$ $\tan(k\_L) = 4.49342$ or ... $k\_L1 := 4.5$

i.e. k^2 = 4.49342^2/L^2 or ... $P_{cr} := 4.49341^2\cdot\frac{E\cdot I}{L^2}$ $\mathrm{root}(\tan(k\_L1) - k\_L1, k\_L1) = 4.4934$

to see in general form multiply and divide by π^2

$P_{cr} := \pi^2\cdot\frac{E\cdot I}{\left(\frac{\pi}{4.49341}\cdot L\right)^2}$ and $\frac{\pi}{4.49341} = 0.6992 \sim 0.7 \Rightarrow P_{cr} := \pi^2\cdot\frac{E\cdot I}{(0.7\cdot L)^2}$

we stated at the introduction to this segment that buckling was a situation where deflection was not proportional to applied force (i.e. general definition of force includes moment). In Euler buckling the deflection is proportional to axial force up to Pcr - note the proportionality is strain in the axial direction. Now let's look at a problem where the axial force is combined with a tranverse force Q (a point force). For simplicity we will locate it at the center of a beam-column so we can use symmetry. see Timoshenko & Gere section 1-3 for an arbitrary placement. (figure later)



In this case the equations are as follows: $M(x) := \frac{Q}{2}\cdot x + P\cdot y$

using: $M(x) := -E\cdot I\cdot\frac{d^2}{dx^2}y(x)$ => $\frac{d^2}{dx^2}\left(y(x) + \frac{P}{E\cdot I}\cdot y(x)\right) = -\frac{Q\cdot x}{2\cdot E\cdot I}$

and as above let k^2 = P/(E*I). leads to solution:

$$y(x) := A\cdot\cos(k\cdot x) + B\cdot\sin(k\cdot x) - \frac{Q\cdot x}{2\cdot P}$$

specific solution:

$y(x) := c\cdot x$

$\frac{P}{E\cdot I}\cdot y(x) \rightarrow \begin{pmatrix} 0 \\ c \end{pmatrix} = -\frac{Q\cdot x}{2\cdot E\cdot I}$

$c := \frac{-Q}{2\cdot P}$ $\quad y(x) := \frac{-Q}{2\cdot P}\cdot x$

the boundary conditions are y(0) = 0 => $A := 0$

and $\frac{d}{dx}y\left(\frac{L}{2}\right) = 0$ => $B := \frac{Q}{2\cdot P\cdot k}\cdot\frac{1}{\cos\left(k\cdot\frac{L}{2}\right)}$ =>

$$y(x) := \frac{Q}{2\cdot P\cdot k}\cdot\frac{\sin(k\cdot x)}{\cos\left(k\cdot\frac{L}{2}\right)} - \frac{Q}{2\cdot P}\cdot x$$

consider deflection at the midpoint (maximum):

$$y\left(\frac{L}{2}\right) = \frac{Q}{2\cdot P\cdot k}\cdot\frac{\sin\left(k\cdot\frac{L}{2}\right)}{\cos\left(k\cdot\frac{L}{2}\right)} - \frac{Q}{2\cdot P}\cdot\frac{L}{2} \quad => \quad y\left(\frac{L}{2}\right) = \frac{Q}{2\cdot P\cdot k}\cdot\left(\tan\left(k\cdot\frac{L}{2}\right) - k\cdot\frac{L}{2}\right)$$

following Timoshenko: let $u := k\cdot\frac{L}{2}$ and using some algebra and

substitutions; $k := \frac{2\cdot u}{L}$; $P := E\cdot I\cdot k^2$; $P := E\cdot I\cdot 4\cdot\frac{u^2}{L^2}$ => $\frac{Q}{2\cdot P\cdot k} = \frac{Q}{2\cdot\left(E\cdot I\cdot 4\cdot\frac{u^2}{L^2}\right)\cdot\frac{2\cdot u}{L}} = \frac{1}{16}\cdot\frac{Q\cdot L^3}{E\cdot I\cdot u^3}$

which finally is = $\frac{1}{48}\cdot\frac{Q\cdot L^3}{E\cdot I}\cdot\frac{3}{u^3}$ => $y\left(\frac{L}{2}\right) = \frac{1}{48}\cdot\frac{Q\cdot L^3}{E\cdot I}\cdot\left[\frac{3}{u^3}\cdot(\tan(u) - u)\right]$

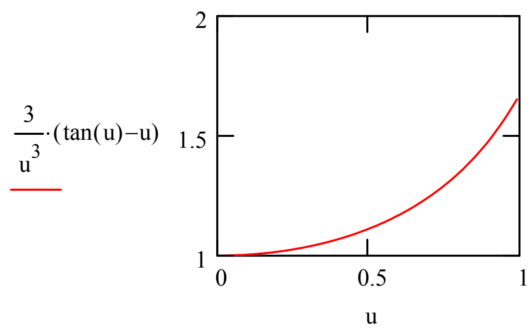
now why did we (Timoshenko) go to all that trouble?

The deflection at L/2 is now in a form $\frac{1}{48}\cdot\frac{Q\cdot L^3}{E\cdot I}$: the deflection due to the force Q times a multiplier

with the following properties: (again with some substitutions) $k := \sqrt{\frac{P}{E\cdot I}}$; $\quad u := \frac{L}{2}\cdot k$; $\quad u := \frac{L}{2}\cdot\sqrt{\frac{P}{E\cdot I}}$

so when P is small ~ 0 ; $\quad \frac{3}{u^3}\cdot(\tan(u) - u) \to 1$ which can be seen by expanding tan(u) in a

series: $\tan(u) = u + \frac{u^3}{3} + \ldots$; $\quad \frac{3}{u^3}\cdot(\tan(u) - u) = \frac{3}{u^3}\cdot\left(u + \frac{u^3}{3} - u\right) \approx 1$ or plotting: $u := 0.001, 0.011 .. 1$



also as u => $\pi/2$ tan(u) => ∞

at this value $u := \frac{L}{2}\cdot\sqrt{\frac{P}{E\cdot I}}$ or $\frac{\pi}{2} = \frac{L}{2}\cdot\sqrt{\frac{P}{E\cdot I}}$

$P := \pi^2\cdot\frac{E\cdot I}{L^2}$ which is the value for P_{cr} above.

using $P_{cr} := \pi^2\cdot\frac{E\cdot I}{L^2}$ in the definition for u =>

$$u := \frac{\pi}{2}\cdot\sqrt{\frac{P}{P_{cr}}}$$

recalling that $y(x) := \frac{Q}{2\cdot P\cdot k}\cdot\frac{\sin(k\cdot x)}{\cos\left(k\cdot\frac{L}{2}\right)} - \frac{Q}{2\cdot P}\cdot x$ and using this same approach: the slope at y = 0

$\frac{d}{dx}y(0) = \frac{Q\cdot L^2}{16\cdot E\cdot I}\cdot\left[\frac{2\cdot(1 - \cos(u))}{u^2\cdot\cos(u)}\right]$ and the maximum moment is $M_{max} := \frac{Q\cdot L}{4}\cdot\left(\frac{\tan(u)}{u}\right)$

these are both in the form of the effect of Q * a multiplier that

=> 1 for u , P small (=> 0) and => ∞ as u => $\pi/2$ or P => P_{cr}

before leaving this approach to buckling, let's consider when q(x) is not 0 or a point but is uniform per unit length:
for a pinned column: see this link

Reference:C:\Documents and Settings\Dave Burke\My Documents\structures\overall_technical\buckling\eqn_11_3_1.mcd(R)

the result is: (for w_{max}) as stated in Hughes equation (11.3.1)

$$w_{max}(\xi) := \frac{5\cdot q\cdot L^4}{384\cdot E\cdot I}\cdot\left[\frac{24}{5\cdot\xi^4}\cdot\left(\sec(\xi) - 1 - \frac{\xi^2}{2}\right)\right] \quad \text{with} \quad \xi := \frac{L}{2}\cdot\sqrt{\frac{P}{E\cdot I}}$$

$$M_{max} := \frac{q\cdot L^2}{8}\cdot\left[\frac{2\cdot(1-\sec(\xi))}{\xi^2}\right]$$

note that ξ is the u above and w is y. The section is titled "use of the magnification factor".

a few more things to deveop for euler and general elastic (and other) buckling:

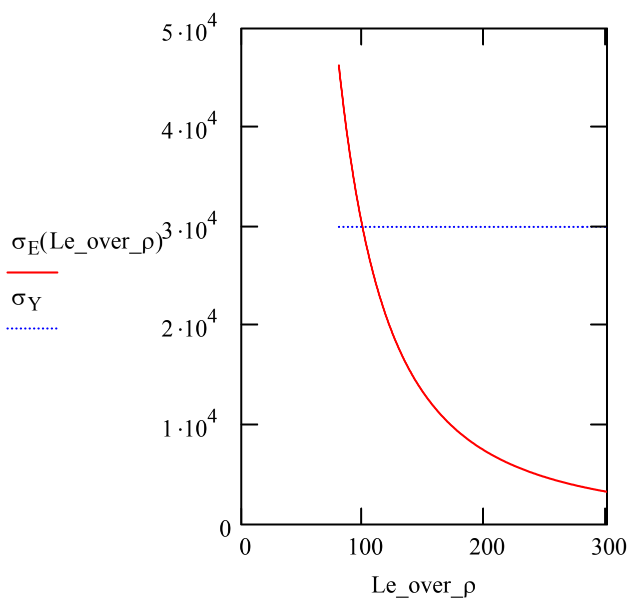
recall the definition of the radius of gyration = ρ defined such that $I := \rho^2\cdot A$ or $\rho := \sqrt{\frac{I}{A}}$ and

the definition of stress is force (P) / area (A).

$$\sigma_{cr} := \frac{P_{cr}}{A} \qquad \sigma_E := \frac{P_{cr}}{A} \qquad P_{cr} := \pi^2\cdot\frac{E\cdot I}{L^2} \qquad \sigma_E := \frac{\pi^2\cdot\frac{E\cdot I}{L^2}}{A} \qquad \sigma_E := \pi^2\cdot\frac{E}{\left(\frac{L}{\rho}\right)^2}$$

a typical plot of euler stress vs L/ρ (slenderness ratio):

$$Le\_over\_\rho := 80, 81 .. 300 \qquad \sigma_Y := 30000 \qquad E := 30\cdot 10^6 \qquad \sigma_E(Le\_over\_\rho) := \frac{\pi^2\cdot E}{Le\_over\_\rho^2}$$



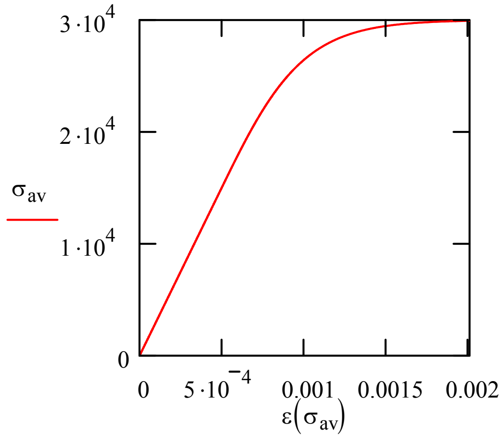
as can be seen, the euler stress is yield when

$$\frac{\pi^2\cdot E}{Le\_over\_\rho^2} = \sigma_Y \text{ or } Le\_over\_\rho := \sqrt{\frac{\pi^2\cdot E}{\sigma_Y}} \text{ i.e.}$$

$$Le\_over\_\rho = 99.3$$

to understand the slenderness ratio better and the difference between slender and "squat" (short fat) columns consider the following example.

Residual stresses from rolling or welding lead to reductions in modulus:

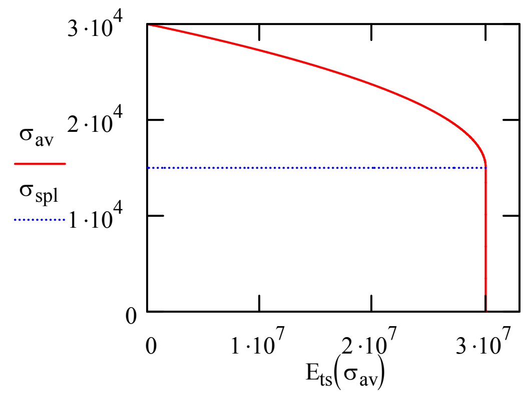


the decrease in $E = \frac{d}{d\varepsilon}\sigma$ approximates a parabolic shape above a value of σ_{av} defined as the structural proportional limit typically $\sigma_{spl} := \frac{\sigma_Y}{2}$ e.g. above the proportional limit:

$$E_{ts}(\sigma_{av}) := \frac{\sigma_{av}\cdot(\sigma_Y - \sigma_{av})}{\sigma_{spl}\cdot(\sigma_Y - \sigma_{spl})}\cdot E$$

redefining because the modulus is reduced only above the proportional limit

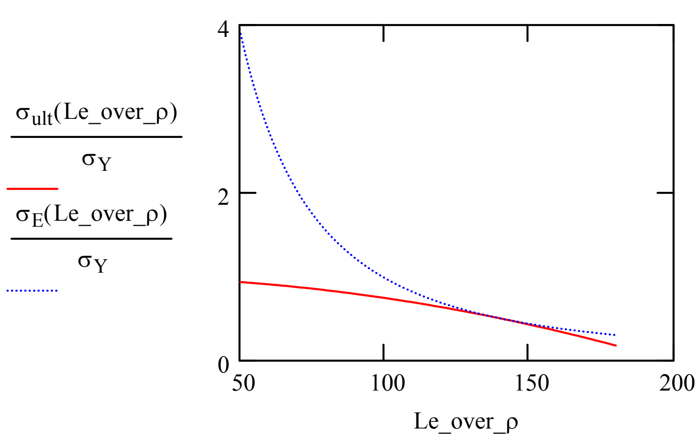
$$E_{ts}(\sigma_{av}) := \text{if}\left[\sigma_{av} > \sigma_{spl}, \frac{\sigma_{av}\cdot(\sigma_Y - \sigma_{av})}{\sigma_{spl}\cdot(\sigma_Y - \sigma_{spl})}\cdot E, E\right]$$



with some algebra and designating σ_{ult} as σ_{av} an expression for σ_{ult} vs. Le_over_ρ results:

$$Le\_over\_\rho := 50..180$$

$$\sigma_{ult}(Le\_over\_\rho) := \left[1 - \frac{\sigma_{spl}}{\sigma_Y}\cdot\left(1 - \frac{\sigma_{spl}}{\sigma_Y}\right)\cdot\frac{\sigma_Y}{\sigma_E(Le\_over\_\rho)}\right]\cdot\sigma_Y$$



recall from above:

$$\sigma_E(Le\_over\_\rho) := \frac{\pi^2\cdot E}{Le\_over\_\rho^2}$$

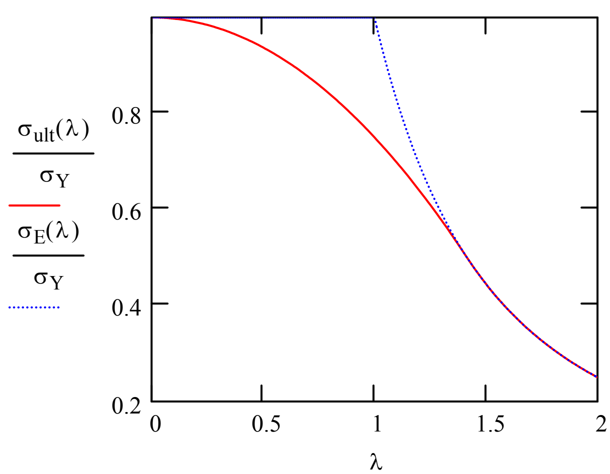
if we define a ratio $\lambda := \sqrt{\frac{\sigma_Y}{\sigma_E}}$ defined as the column slenderness parameter which appears above (as λ^2) => $\sigma_E(\lambda) := \frac{\sigma_Y}{\lambda^2}$

σ_{ult} becomes $\sigma_{ult}(\lambda) := \left[1 - \frac{\sigma_{spl}}{\sigma_Y} \cdot \left(1 - \frac{\sigma_{spl}}{\sigma_Y}\right) \cdot \lambda^2\right] \cdot \sigma_Y$

applying when $\sigma_{spl} < \sigma_{ult} < \sigma_Y$, and limiting σ_E to σ_Y =>

$\sigma_E(\lambda) := \min\left(\begin{pmatrix} \sigma_Y \\ \frac{\sigma_Y}{\lambda^2} \end{pmatrix}\right)$ => $\sigma_{ult}(\lambda) := \mathrm{if}\left[\lambda \le \sqrt{2}, \left[1 - \frac{\sigma_{spl}}{\sigma_Y} \cdot \left(1 - \frac{\sigma_{spl}}{\sigma_Y}\right) \cdot \lambda^2\right] \cdot \sigma_Y, \frac{\sigma_Y}{\lambda^2}\right]$

$\lambda := 0.0, 0.01 .. 2$

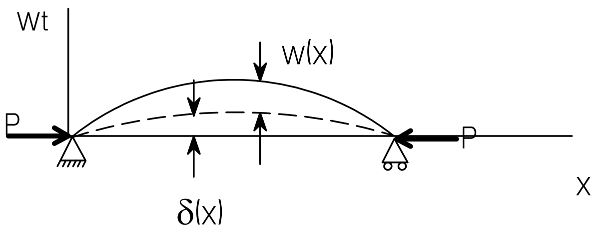


reduces to: $1 - \frac{\lambda^2}{4}$

when σ_{spl}/σ_Y = 0.5 as shown here.

$P_E = \pi^2 \cdot \frac{E \cdot I}{L^2}$ $\qquad \rho^2 = \frac{I}{A}$ $\qquad \frac{P_E}{A} = \sigma_E = \pi^2 \cdot \frac{E \cdot I}{L^2} \cdot \frac{1}{A} = \pi^2 \cdot \frac{E \cdot \frac{I}{A}}{L^2} = \pi^2 \cdot \frac{E \cdot \rho^2}{L^2} = \pi^2 \cdot \frac{E}{\left(\frac{L}{\rho}\right)^2} = \sigma_E$

other factors that affect column behavior are not being perfectly straight and application of the load off center these are termed eccentricity in geometry and load application. Consider first geometry: for a column with an initial deflection $\delta(x)$ (ref: Hughes pp 394 ff):



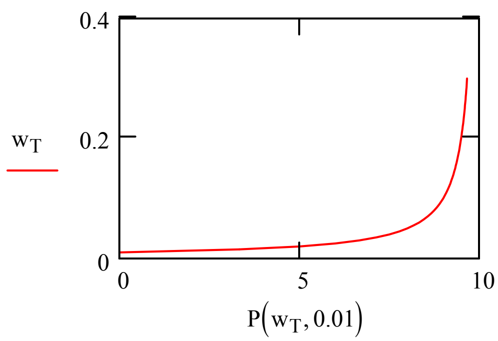
$$\frac{d^2}{dx^2}w + \frac{P}{E \cdot I} \cdot (\delta + w) = 0$$

assuming a sinusoidal (Fourier series) deflection results in a deflection $w_T := \frac{P_E}{P_E - P} \cdot \delta$

this has the property we saw earlier, a result with a magnification factor ϕ as
$w_T := \phi \cdot \delta$.

When P is small the deflection matches δ. When P approaches the euler value, the deflection is very large. The deflection is continuous, not proportional to P. The load deflection curves for eccentric columns are shown

$P_E := 10$ $\qquad$ $w_T := 0.01, 0.015 .. 0.3$ $\qquad$ $P(w_T, \delta) := P_E \cdot \left(1 - \frac{\delta}{w_T}\right)$



total displacement vs. applied force P

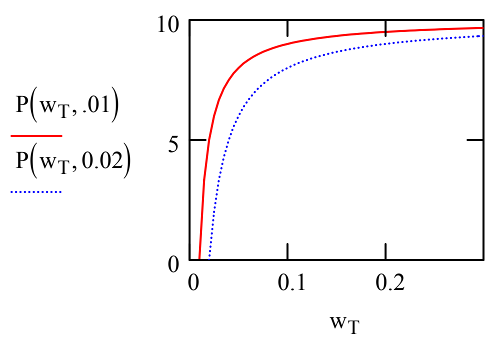
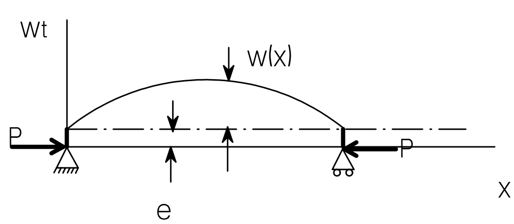


figure 11.8

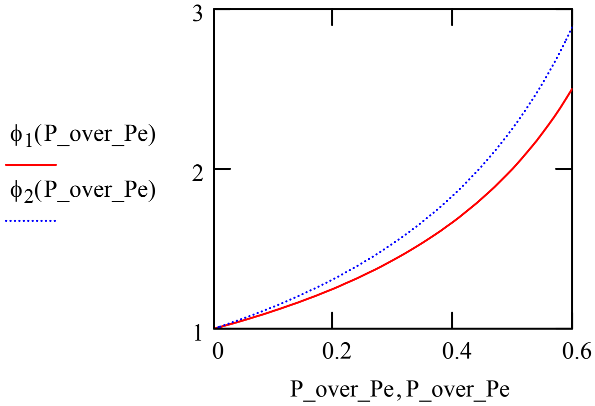
eccentricity in load application is derived in Timoshenko with the same form with different magnification factor:

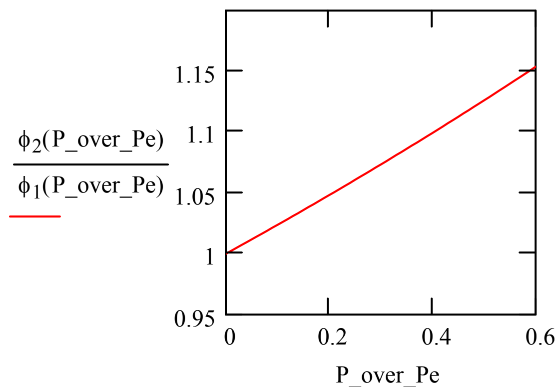
$\phi := \sec\left(\frac{\pi}{2} \cdot \sqrt{\frac{P}{P_E}}\right)$ $\qquad$ $M := P \cdot e \cdot \phi$:



This factor approximates the factor for geometry, as can be seen from the following plots so it is simpler to use the geometry magnification factor for both effects

$P\_over\_Pe := 0, 0.01 .. 0.6 \quad \phi_1(P\_over\_Pe) := \dfrac{1}{1 - P\_over\_Pe} \qquad \phi_2(P\_over\_Pe) := \sec\left(\dfrac{\pi}{2}\cdot\sqrt{P\_over\_Pe}\right)$

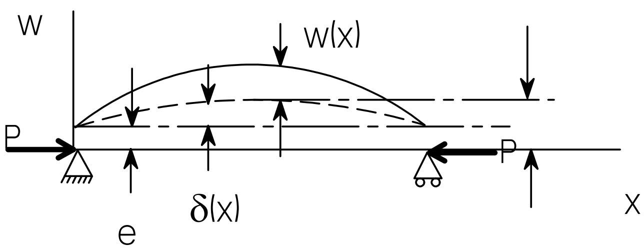




this allows us to combine the two eccentricities δ (geometry) and e (off center load) such that $\Delta := \delta + e$

the "magnified" deflection becomes $\Delta^*\phi$ the moment from the applied force P then becomes $M := P\cdot\Delta\cdot\phi$

the total stress in a column accounting for both compression and bending is then:



$\sigma_{max} := \dfrac{P}{A} + \dfrac{P\cdot\Delta\cdot\phi}{Z}$ where Z is the modulus in the direction (extreme fiber) that undergoes compression. this

expression can be rearranged as follows: defining $r_c := \dfrac{Z}{A} = \dfrac{I}{c}\cdot\dfrac{1}{A} = \dfrac{\rho^2}{c}$ where c is the distance of the extreme fiber (compression side) to the neutral axis.

$\sigma_{max} := \dfrac{P}{A}\cdot\left(1 + \dfrac{\Delta\cdot\phi}{\dfrac{Z}{A}}\right) = \sigma_{max} := \dfrac{P}{A}\cdot\left(1 + \dfrac{\Delta}{r_c}\cdot\phi\right)$ where $\dfrac{\Delta}{r_c}$ = eccentricity ratio.

estimates for eccentricity ratio have been accepted based on experimental evidence as proportional to slenderness ratio $\dfrac{L_e}{\rho}$ i.e. $\dfrac{\Delta}{r_c} = \alpha\cdot\dfrac{L_e}{\rho}$ and σ_{max} becomes $\sigma_{max} := \dfrac{P}{A}\cdot\left(1 + \alpha\cdot\dfrac{L_e}{\rho}\cdot\phi\right)$

if we now replace ϕ by $\phi := \frac{P_E}{P_E - P}$; designate P as P_{ult} and declare "failure" when $\sigma_{max} = \sigma_Y$

we obtain $\sigma_Y := \frac{P_{ult}}{A} \cdot \left(1 + \frac{\alpha \cdot \frac{L_e}{\rho} \cdot P_E}{P_E - P_{ult}} \right)$ eqn 11.2.1 rearranged => $\sigma_Y := \frac{P_{ult}}{A} \cdot \left(1 + \frac{\alpha \cdot \frac{L_e}{\rho}}{1 - \frac{P_{ult}}{P_E}} \right)$

or in terms of stress $\sigma_Y := \sigma_{ult} \cdot \left(1 + \frac{\alpha \cdot \frac{L_e}{\rho}}{1 - \frac{\sigma_{ult}}{\sigma_E}} \right)$ eqn 11.2.2 σ_{ult} is the applied stress that will result in yield after accounting for the magnification factor as developed above.

if we define $R := \frac{\sigma_{ult}}{\sigma_Y}$; $\eta := \frac{\alpha \cdot L}{\rho}$; and use the column slenderness parameter $\lambda := \sqrt{\frac{\sigma_Y}{\sigma_E}}$ or $\lambda := \frac{L}{\pi \cdot \rho} \cdot \sqrt{\frac{\sigma_Y}{E}}$ to solve eqn 11.2.2 above for $R := \frac{\sigma_{ult}}{\sigma_Y}$ we obtain a quadratic equation for R

$(1 - R) \cdot \left(1 - \lambda^2 \cdot R\right) = \eta \cdot R$ with solution (taking the negative sign in the quadratic term) and a lot of algebra:

the Perry Robertson column formula results:

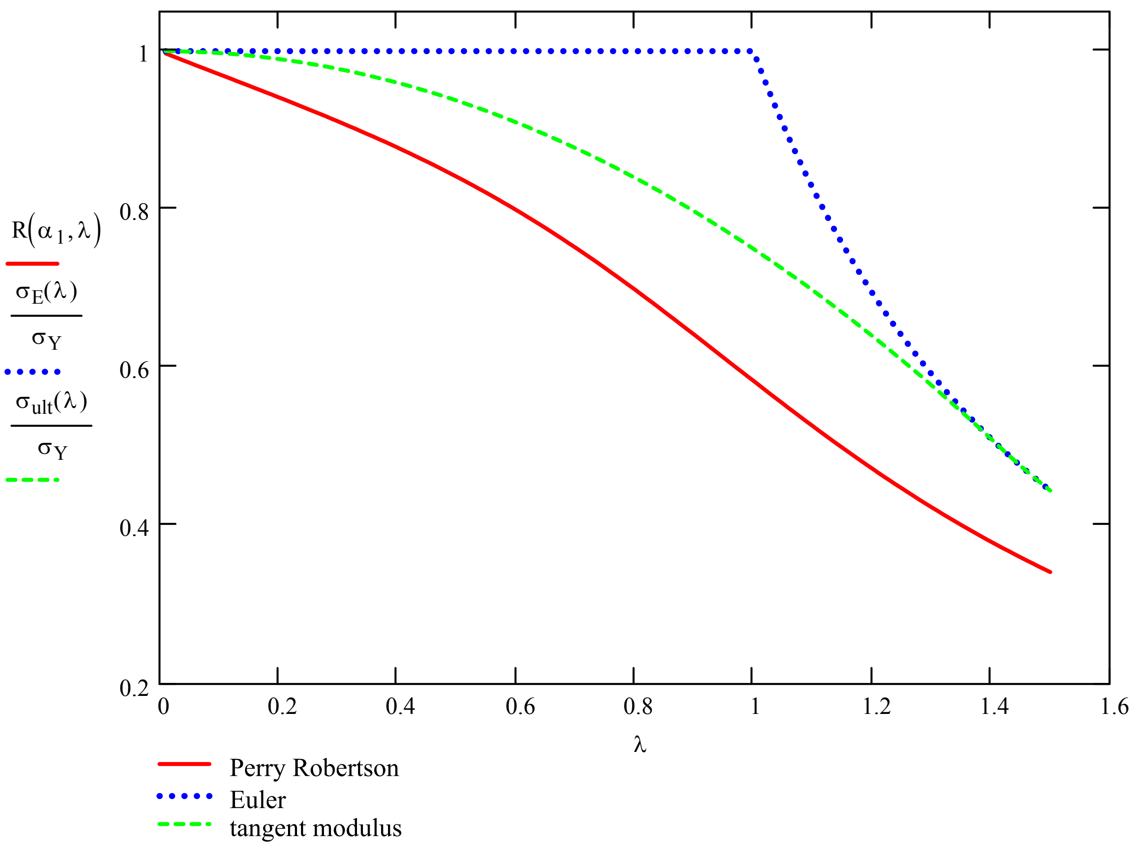
$\lambda := 0.01, 0.02 .. 1.5$ $\alpha_1 := 0.003$ $\eta(\alpha, \lambda) := \left(\alpha \cdot \pi \cdot \sqrt{\frac{E}{\sigma_Y}} \right) \cdot \lambda$

$$R(\alpha, \lambda) := \left[\frac{1}{2} \cdot \left(1 + \frac{1 + \eta(\alpha, \lambda)}{\lambda^2} \right) - \sqrt{\frac{1}{4} \cdot \left(1 + \frac{1 + \eta(\alpha, \lambda)}{\lambda^2} \right)^2 - \frac{1}{\lambda^2}} \right]$$

as above euler = $\sigma_E(\lambda) := \min\left(\begin{pmatrix} \sigma_Y \\ \frac{\sigma_Y}{\lambda^2} \end{pmatrix} \right)$ and tangent modulus

$$\sigma_{ult}(\lambda) := \mathrm{if}\left[\lambda \le \sqrt{2}, \left[1 - \frac{\sigma_{spl}}{\sigma_Y} \cdot \left(1 - \frac{\sigma_{spl}}{\sigma_Y} \right) \cdot \lambda^2 \right] \cdot \sigma_Y, \frac{\sigma_Y}{\lambda^2} \right]$$

comparison of these different approaches =>



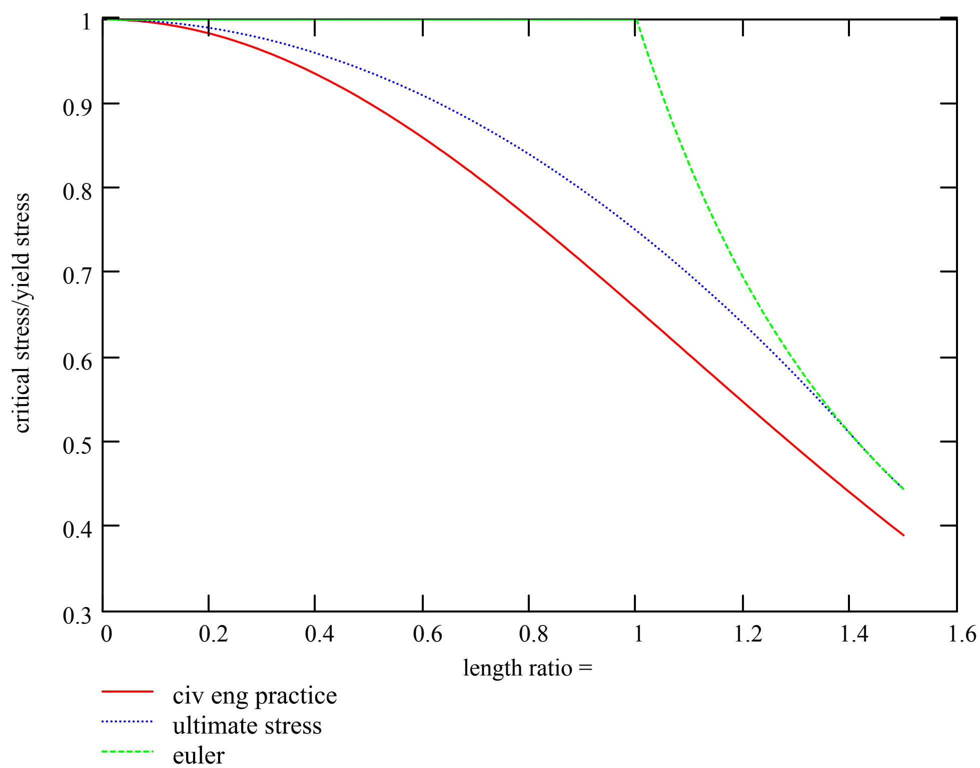
this is CCB taking $R(\alpha,\lambda) := \dfrac{\sigma_{ult}(\alpha,\lambda)}{\sigma_Y}$ with $\alpha := .002$ from Table 11.1 and assuming $L_e := .7\cdot L_{col}$ in calculating λ and $\eta := \alpha\cdot\left(\dfrac{L_{col}}{\rho}\right)$ σ_{ult} is the applied stress that will result in yield after accounting for the magnification factor

$\sigma_{ult} = R\cdot\sigma_Y$ $\qquad \sigma_a := \dfrac{P}{A}$ $\qquad \gamma R_{CCB} := \gamma_C\cdot\left(\dfrac{\sigma_a}{\sigma_{ult}}\right)$

Page 322 S&J: The civil engineers use a curve similar to this for column design Table in manual of steel construction:
the loads include a resistance factor (1/PSF) of 0.85. From LRFD spec E-2. Curve 2 fit to out-of-straightness =1/1500.

SSRC guide S&J pg 322

$$\sigma_c(\lambda) := \mathrm{if}\left(\lambda \le 1.5, 0.658^{\lambda^2}\cdot\sigma_Y, \frac{0.877}{\lambda^2}\cdot\sigma_Y\right)$$



Check using column table pg 2-35 Manual of Steel Construction (MSC)
KL=10ft, nominal diameter = 10 in, extra strong. ϕ = resistance factor. Load = 465,000 lbs

redefining values for MSC

$l_e := 10\cdot 12$ $\quad A := 16.1$ $\quad \rho := 3.63$ $\quad \phi := 0.85$ $\quad \sigma_Y := 36000$ $\quad E := 29000000$

$$\lambda := \frac{l_e}{\rho}\cdot\sqrt{\frac{\sigma_Y}{\pi^2\cdot E}} \qquad \lambda = 0.3707$$

SSRC guide S&J pg 322

$$\sigma_c(\lambda) := \mathrm{if}\left(\lambda \le 1.5, 0.658^{\lambda^2}\cdot\sigma_Y, \frac{0.877}{\lambda^2}\cdot\sigma_Y\right) \qquad P(\lambda) := \sigma_c(\lambda)\cdot A\cdot\phi$$

$P(\lambda) = 465117$ compares to 465 ksi in table

reset yield, modulus and curve to general values

$\sigma_Y := 30000$ $\qquad E := 30\cdot 10^6$

$$\sigma_c(\lambda) := \mathrm{if}\left(\lambda \le 1.5, 0.658^{\lambda^2}\cdot\sigma_Y, \frac{0.877}{\lambda^2}\cdot\sigma_Y\right)$$

Accurate design curves, figures 11.11, 11.12, 11.13

define E to SI units per text

$E := 200000$

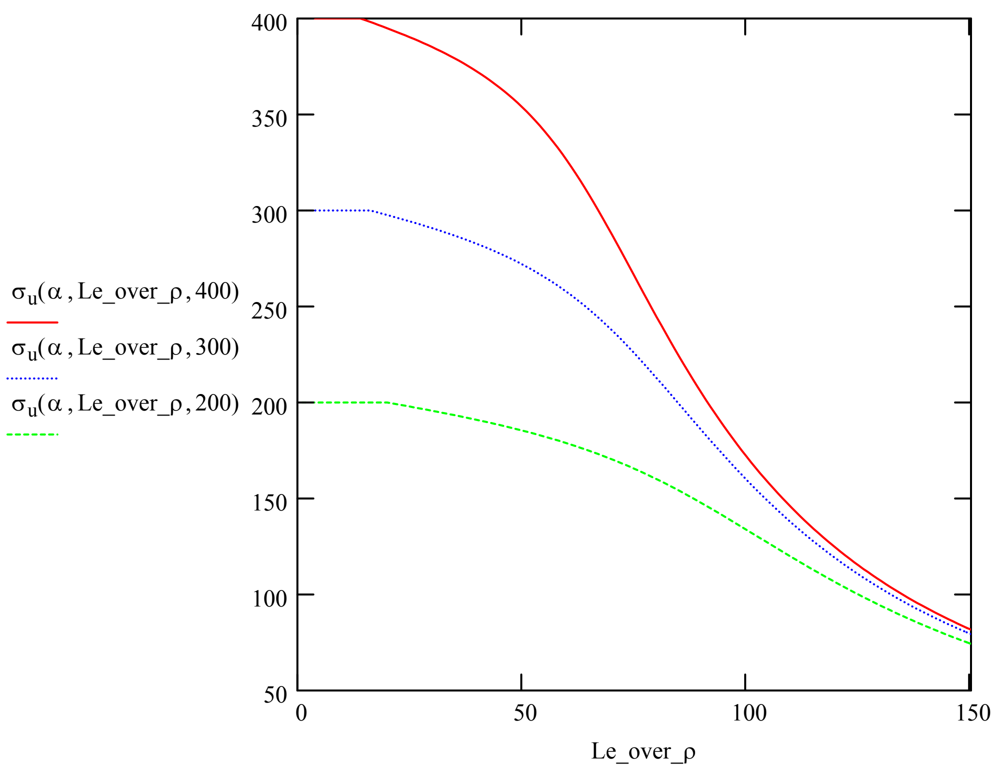
$\alpha := 0.002$ choose α based on column shape, α = 0.002 for circular

$Le\_over\_\rho := 1..150$

$$\lambda(Le\_over\_\rho, \sigma_Y) := \frac{Le\_over\_\rho}{\pi} \cdot \sqrt{\frac{\sigma_Y}{E}}$$

$$\eta(\alpha, Le\_over\_\rho, \sigma_Y) := \text{if}\left[\lambda(Le\_over\_\rho, \sigma_Y) \le 0.2, 0, \left(\alpha \cdot \pi \cdot \sqrt{\frac{E}{\sigma_Y}}\right) \cdot (\lambda(Le\_over\_\rho, \sigma_Y) - 0.2)\right]$$

$$\sigma_u(\alpha, Le\_over\_\rho, \sigma_Y) := \left[\frac{1}{2} \cdot \left(1 + \frac{1 + \eta(\alpha, Le\_over\_\rho, \sigma_Y)}{\lambda(Le\_over\_\rho, \sigma_Y)^2}\right) - \sqrt{\frac{1}{4} \cdot \left(1 + \frac{1 + \eta(\alpha, Le\_over\_\rho, \sigma_Y)}{\lambda(Le\_over\_\rho, \sigma_Y)^2}\right)^2 - \frac{1}{\lambda(Le\_over\_\rho, \sigma_Y)^2}}\right]$$



$\sigma_u(\alpha, 90, 200) = 147.5465$

text has ~ 150 fig 11.11

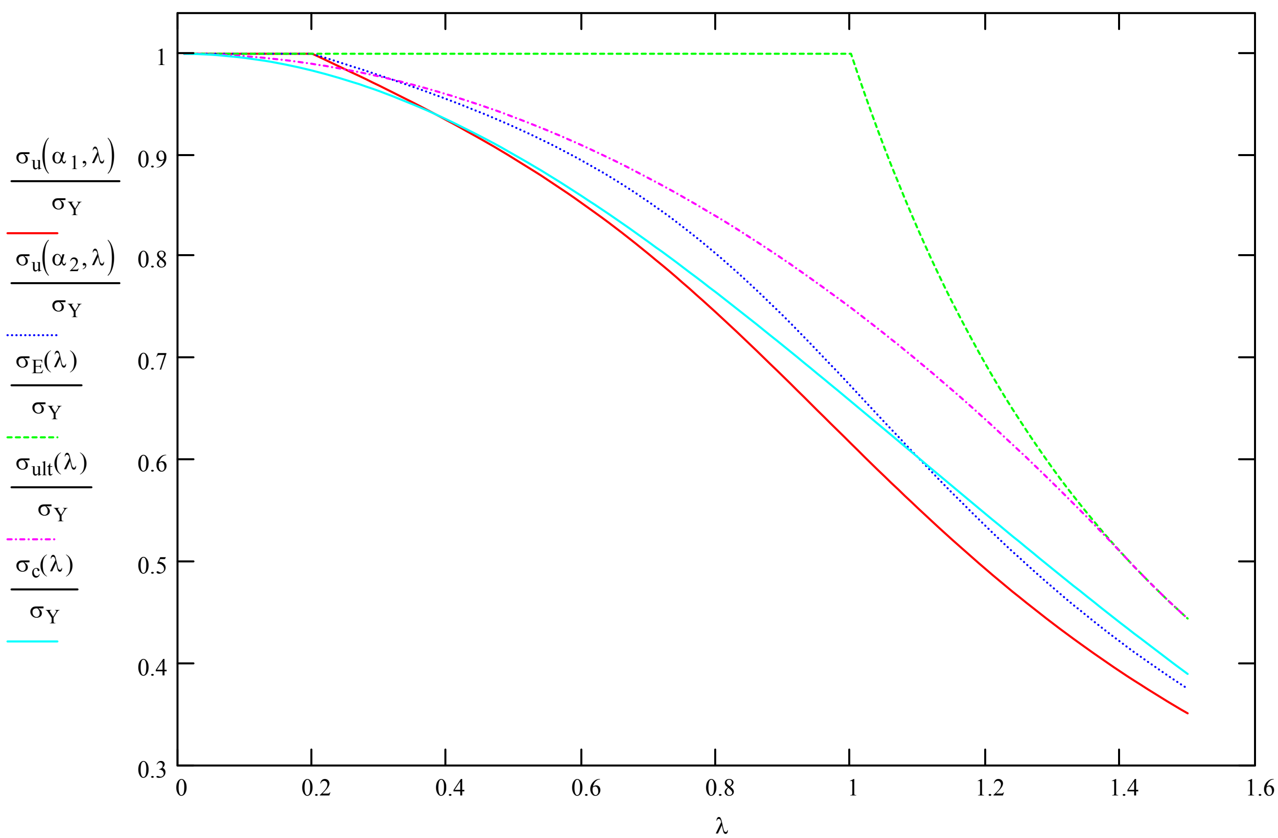
revisiting the Perry Robertson relationship with λ offset

redefine E to Emglish units

$\lambda := 0.01, 0.02 .. 1.5$ $\alpha_1 := 0.003$ $\alpha_2 := 0.002$ $E := 30\cdot 10^6$

$$\eta(\alpha,\lambda) := \text{if}\left[\lambda \le 0.2, 0, \left(\alpha\cdot\pi\cdot\sqrt{\frac{E}{\sigma_Y}}\right)\cdot(\lambda - 0.2)\right]$$

$$\sigma_u(\alpha,\lambda) := \left[\frac{1}{2}\cdot\left(1 + \frac{1+\eta(\alpha,\lambda)}{\lambda^2}\right) - \sqrt{\frac{1}{4}\cdot\left(1 + \frac{1+\eta(\alpha,\lambda)}{\lambda^2}\right)^2 - \frac{1}{\lambda^2}}\right]\cdot\sigma_Y$$



Timoshenko Th of Elas Stab sect 1.7

$$\frac{1}{1-\frac{P}{P_{cr}}}$$

can be used as approximation of all amplification factors, $\chi(u)$, $\eta(u)$ and $\lambda(u)$ good for P/Pe<0.6

$$u := \frac{k\cdot l}{2} \qquad k\_sq := \frac{P}{E\cdot I} \qquad P_{cr} := \frac{\pi^2\cdot E\cdot I}{l^2} \qquad P\_over\_Pcr := \frac{k^2\cdot E\cdot I}{\pi^2\cdot E\cdot I}\cdot l^2 \qquad P\_over\_Pcr := \left(\frac{k\cdot l}{\pi}\right)^2$$

$$u := \frac{\pi}{2}\cdot\sqrt{\frac{P}{P_{cr}}} \qquad P\_over\_Pcr := \left(\frac{2\cdot u}{\pi}\right)^2 \qquad P\_over\_Pcr(u) := \left(\frac{2\cdot u}{\pi}\right)^2$$

$$\phi(u) := \frac{1}{1-\left(\frac{2\cdot u}{\pi}\right)^2}$$

or using u=k*l/2 and k=sqrt(P/EI) and EI=P_{cr}*l^2/π^2

$$u := \frac{\pi}{2}\cdot\sqrt{\frac{P}{P_{cr}}}$$

amplification factor for:simply supported,

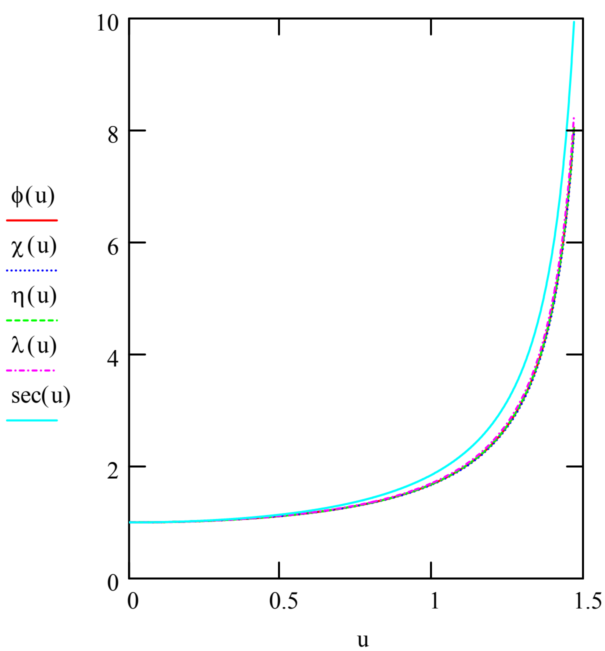
Q at center 1-14 | uniform distribution q 1-21 | couple at ends 1-33 | moment for couple

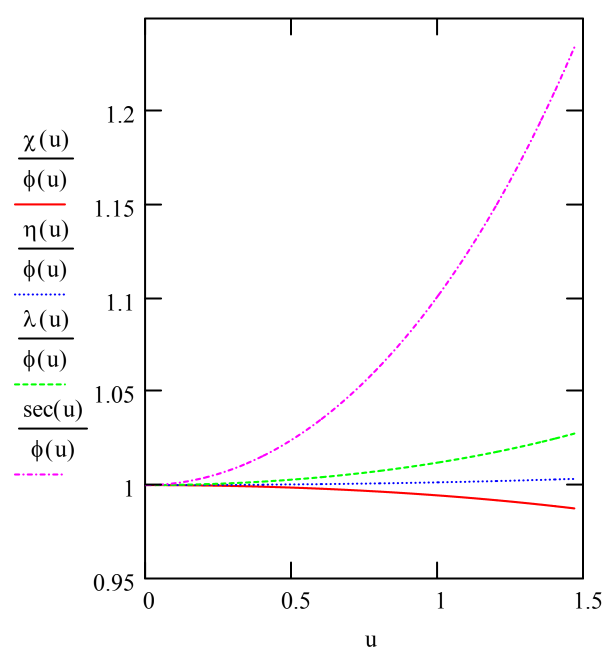
$$\chi(u) := \frac{3\cdot(\tan(u)-u)}{u^3} \qquad \eta(u) := \frac{12\cdot\left(2\cdot\sec(u)-2-u^2\right)}{5\cdot u^4} \qquad \lambda(u) := \frac{2\cdot(1-\cos(u))}{u^2\cdot\cos(u)} \qquad \sec(u)$$

$$u := 0.001, 0.002 .. \frac{\pi}{2} - 0.1$$

$$\frac{\chi(u)}{\phi(u)}, \frac{\eta(u)}{\phi(u)}, \frac{\lambda(u)}{\phi(u)}, \frac{\sec(u)}{\phi(u)}$$

notice scale



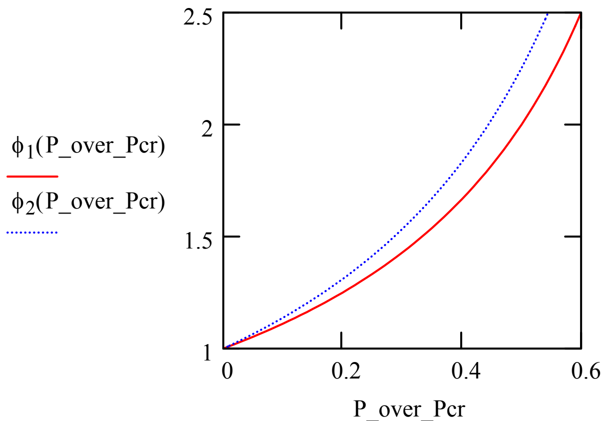


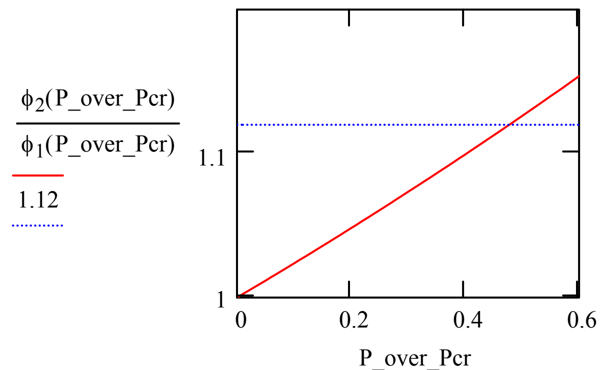
$P\_over\_Pcr := 0, 0.01 .. 0.6$

$\phi_1(P\_over\_Pcr) := \dfrac{1}{1 - P\_over\_Pcr}$

$\phi_2(P\_over\_Pcr) := \sec\left(\dfrac{\pi}{2}\cdot\sqrt{P\_over\_Pcr}\right)$

fig 1-9 T&G





< 12% variation in text for P_over_Pcr < 0.5

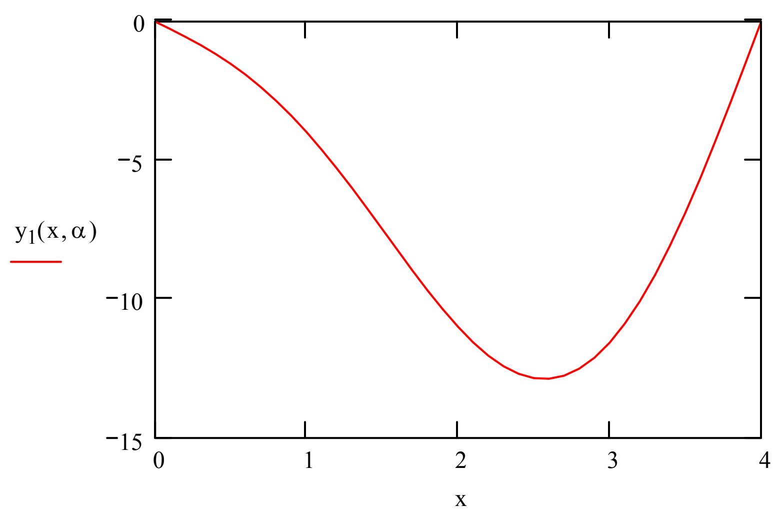
$l := 4 \quad a_1 := 1 \quad a_2 := 10$

$\alpha := 1.1$

watch curve change shape as α -> 1. This is result if second term dominates initial shape. i.e. y_1 has two terms.

$x := 0, 0.1 .. l$

$$y_1(x, \alpha) := \frac{\alpha \cdot a_1}{1 - \alpha} \cdot \sin\left(\frac{\pi \cdot x}{l}\right) + \frac{\alpha \cdot a_2}{2^2 - \alpha} \cdot \sin\left(\frac{2 \cdot \pi \cdot x}{l}\right)$$



Beam column notes $u := \frac{1}{2}\cdot\sqrt{\frac{P}{E\cdot I}}$

$q := 1 \quad P := 1 \quad I := 1$
$M_0 := 1 \quad \phi := 1$
$\Delta := 0$

(at x= l/2) $w_{max}(u) := \frac{5\cdot q\cdot l^4}{384\cdot E\cdot I}\cdot\left[\frac{24}{5\cdot u^2}\cdot\left(\sec(u) - 1 - \frac{u^2}{2}\right)\right]$

$M_{max}(u) := \frac{q\cdot l^2}{8}\cdot\left[\frac{2\cdot(1-\sec(u))}{u^2}\right]$ (11.3.2) see note below re:sign

$M_{max} := M_0 + P\cdot\phi\cdot(w_{max}(u) + \Delta)$

Δ = total eccentricity as in column

comparison of magnification factors;

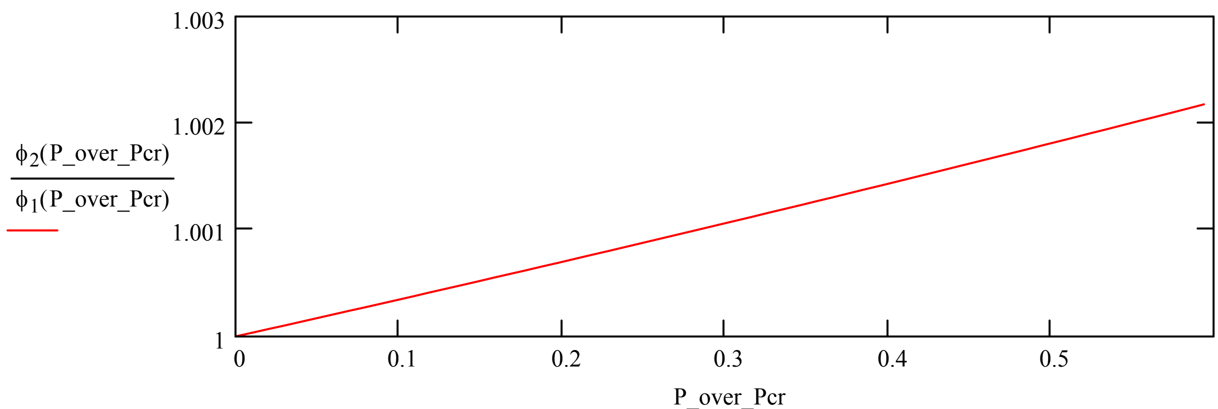
using u=k*l/2 and k=sqrt(P/EI) and EI=P_{cr}*l^2/π^2

$P\_over\_Pcr := 0.0001, 0.01 .. 0.6$

$\phi_1(P\_over\_Pcr) := \frac{1}{1 - P\_over\_Pcr}$

$u(P\_over\_Pcr) := \frac{\pi}{2}\cdot\sqrt{P\_over\_Pcr}$

$$\phi_2(P\_over\_Pcr) := \frac{24\cdot\left(\sec(u(P\_over\_Pcr)) - 1 - \frac{u(P\_over\_Pcr)^2}{2}\right)}{5\cdot u(P\_over\_Pcr)^4}$$



$M_{max1}(P\_over\_Pcr) := M_0\cdot\left[\frac{2\cdot(\sec(u(P\_over\_Pcr)) - 1)}{u(P\_over\_Pcr)^2}\right]$

sign is reversed in text. see P&G eqn 1-23

$M_{max2}(P\_over\_Pcr) := M_0 + P\cdot\phi\cdot(w_{max}(u(P\_over\_Pcr)) + \Delta)$

which after using wmax and P = P_over_Pcr * Pcr becomes with Δ = 0

$M_{max2}(P\_over\_Pcr) := M_0\cdot\left(1 + P\_over\_Pcr\cdot\phi_1(P\_over\_Pcr)\cdot\pi^2\cdot\frac{5}{48}\right)$

$P\_over\_Pcr := 0.5 \qquad \phi_1(P\_over\_Pcr) = 2$

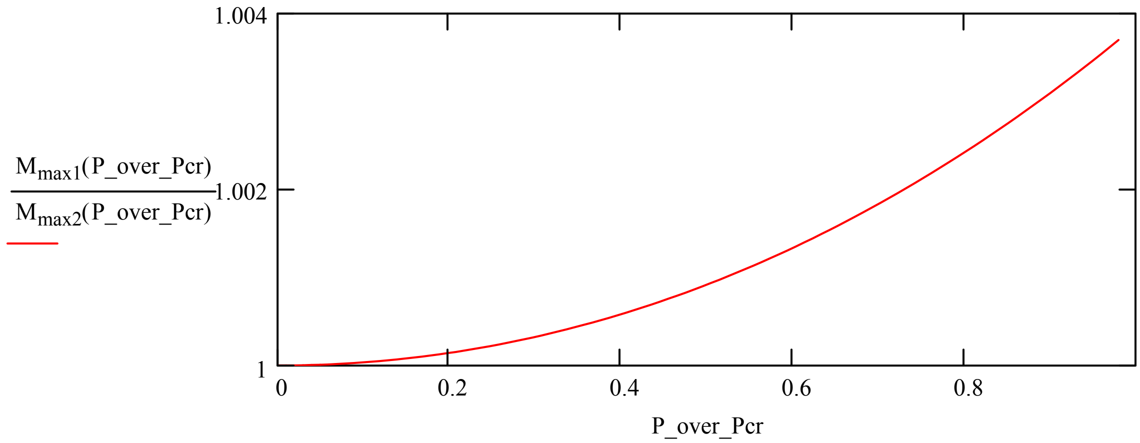
analytical

using ϕ estimate

$M_{max1}(P\_over\_Pcr) = 2.0299 \qquad M_{max2}(P\_over\_Pcr) = 2.0281 \qquad M_{max2}(P\_over\_Pcr = 0.5) = M_0\cdot\left(1 + \frac{5}{48}\cdot\pi^2\right)$

comparison of analytical vs. M due to bending plus P * displacement (magnified)

$P\_over\_Pcr := 0.0001, 0.01 .. 0.99$



This assumption regarding magnification factor ϕ allows using a relationship similar to the Perry-Robertson above with some additional terms to account for the bending moment and displacement (magnified) due to the transverse loading. This is combined with the eccentricity due to the column eccentricity

$M_{max} := M_0 + P\cdot\phi\cdot(\delta_0 + \Delta)$ $\qquad \sigma_{max} := \frac{P}{A} + \frac{M_{max}}{Z}$ $\qquad$ using $\qquad \phi_1(P\_over\_Pcr) := \frac{1}{1 - P\_over\_Pcr}$ $\qquad \sigma_{max} := \sigma_Y$

=> $\sigma_Y = \frac{P_{ult}}{A} + \frac{M_0}{Z} + \frac{P_{ult}\cdot(\delta_0 + \Delta)}{\left(1 - \frac{P_{ult}}{P_E}\right)\cdot Z}$

which after rearranging and defining some non-dimensional factors becomes:

values due to uniform distributed loading (pressure*breadth).

$$\sigma_u(\lambda, \eta, \mu) := \left[\frac{1}{2}\cdot\left(1 - \mu + \frac{1+\eta}{\lambda^2}\right) - \sqrt{\frac{1}{4}\cdot\left(1 - \mu + \frac{1+\eta}{\lambda^2}\right)^2 - \frac{1-\mu}{\lambda^2}}\right]\cdot\sigma_Y$$

$\delta_0 := \frac{5\cdot q\cdot l^4}{384\cdot E\cdot I} \qquad M_0 := \frac{q\cdot l^2}{8}$

with $\qquad \lambda(Le\_over\_\rho) := \sqrt{\frac{\sigma_Y}{\sigma_E}}$

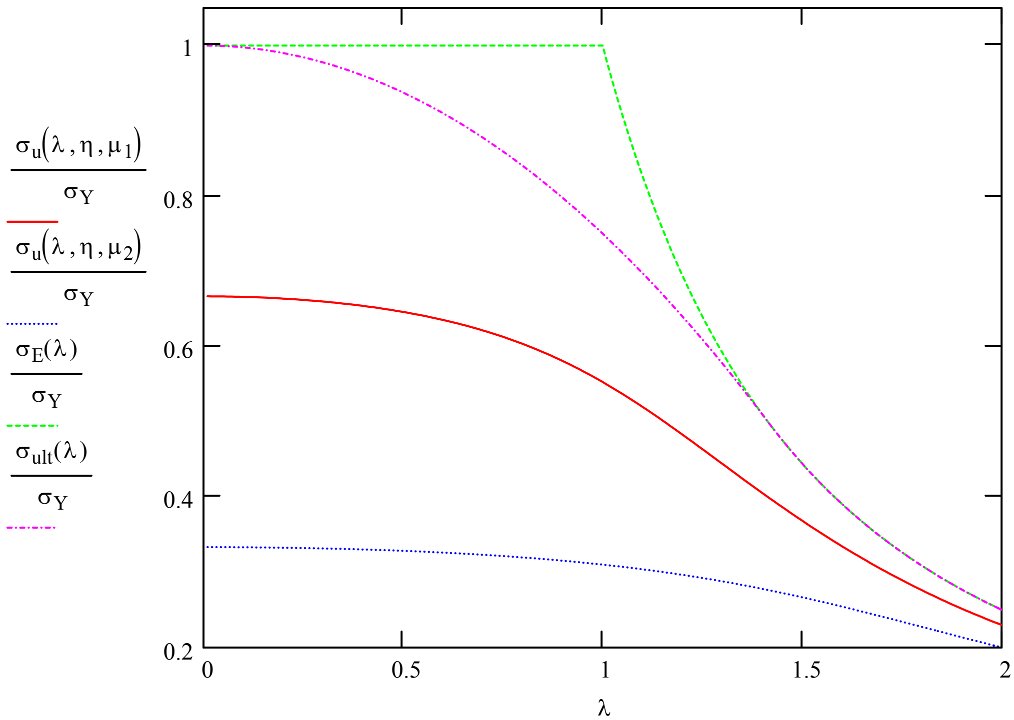
r_c = core radius

$r_c := \frac{\rho^2}{c} \qquad r_c := \frac{I}{A\cdot c} \qquad \mu = \frac{\left(\frac{M_0}{Z}\right)}{\sigma_Y} = \frac{M_0}{Z\cdot\sigma_Y}$

$\lambda(Le\_over\_\rho) := \frac{Le\_over\_\rho}{\pi}\cdot\sqrt{\frac{\sigma_Y}{E}}$ $\qquad \eta := \frac{(\delta_0 + \Delta)}{Z}\cdot A \qquad \eta := \alpha\cdot Le\_over\_\rho + \frac{\delta_0}{r_c}$

figure 11.14 is parameterized by η and μ

$\lambda := 0.01, 0.02 .. 2$ $\eta := 0.2$ $\mu_1 := 0.2$ $\mu_2 := 0.6$



a few values from text to check. To avoid singularity in mathcad, λ set close to 0

$\lambda_0 := 0.0001$

text has 0.83, 0.57, 0.38, 0.29

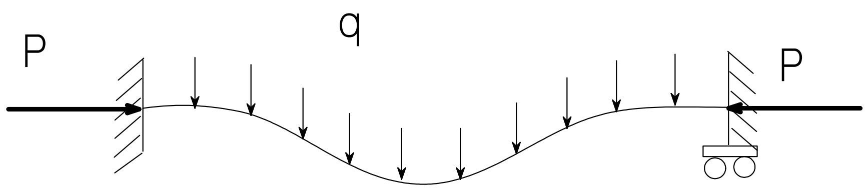
$\frac{\sigma_u(\lambda_0, 0.2, 0)}{\sigma_Y} = 0.8333$ $\frac{\sigma_u(\lambda_0, 0.6, 0.4)}{\sigma_Y} = 0.375$

$\frac{\sigma_u(\lambda_0, 0.4, 0.2)}{\sigma_Y} = 0.5714$ $\frac{\sigma_u(\lambda_0, 0.4, 0.6)}{\sigma_Y} = 0.2857$

Beam Column (page 401 of text)

this was copied from notes 28 beam column and added here

clamped clamped beam column



reset:

$$M(x) := -q\cdot\left(\frac{x^2}{2} - \frac{L\cdot x}{2} + \frac{L^2}{12}\right) \qquad y(x) := \frac{q}{E\cdot I}\cdot\left(\frac{x^4}{24} - \frac{L\cdot x^3}{12} + \frac{L^2\cdot x^2}{24}\right)$$

$$M(0) \rightarrow \frac{-1}{12}\cdot q\cdot L^2 \qquad y\left(\frac{L}{2}\right) \rightarrow \frac{1}{384}\cdot\frac{q}{E\cdot I}\cdot L^4 \qquad \delta_o := y\left(\frac{L}{2}\right) \qquad \delta_o \rightarrow \frac{1}{384}\cdot\frac{q}{E\cdot I}\cdot L^4$$

$$M\left(\frac{L}{2}\right) \rightarrow \frac{1}{24}\cdot q\cdot L^2 \qquad M_{center} := \frac{1}{24}\cdot q\cdot L^2$$

$$M(L) \rightarrow \frac{-1}{12}\cdot q\cdot L^2 \qquad M_{end} := \frac{-1}{12}\cdot q\cdot L^2$$

moments at center and ends

==

find locations where M(x) = 0

$$\text{Given} \quad M(x) = 0 \qquad \text{Find}(x) \rightarrow \left[\left(\frac{1}{2} + \frac{1}{6}\cdot 3^{\frac{1}{2}}\right)\cdot L \quad \left(\frac{1}{2} - \frac{1}{6}\cdot 3^{\frac{1}{2}}\right)\cdot L\right] \qquad x_1 := \left(\frac{1}{2} + \frac{1}{6}\cdot\sqrt{3}\right)\cdot L \qquad x_2 := \left(\frac{1}{2} - \frac{1}{6}\cdot\sqrt{3}\right)\cdot L$$

distance between M(x) = 0 $\quad x_1 - x_2 \text{ simplify} \rightarrow \frac{1}{3}\cdot L\cdot 3^{\frac{1}{2}}$

$$\nu := \frac{1}{\sqrt{3}}\cdot L \qquad \frac{1}{\sqrt{3}} = 0.5774 \qquad \nu := 0.577$$

max deflection (at center) $\quad \delta_o := y\left(\frac{L}{2}\right) \qquad \delta_o \rightarrow \frac{1}{384}\cdot\frac{q}{E\cdot I}\cdot L^4 \qquad \delta_o := \frac{1}{384}\cdot\frac{q}{E\cdot I}\cdot L^4$

deflection at M(X) = 0 $\quad \frac{y(x_1)}{\delta_o} \text{ expand} \rightarrow \frac{4}{9} \qquad \frac{4}{9} = 0.4444 \qquad \gamma := 0.444$

observations of clamped/clamped beam due to uniform load

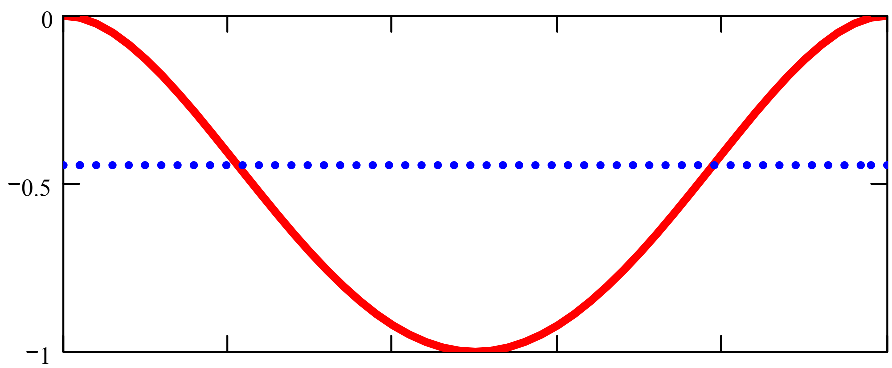
fraction of length between M(x) = 0 $\nu := 0.577$

moment at center $M_{center} := \frac{1}{24} \cdot q \cdot L^2$

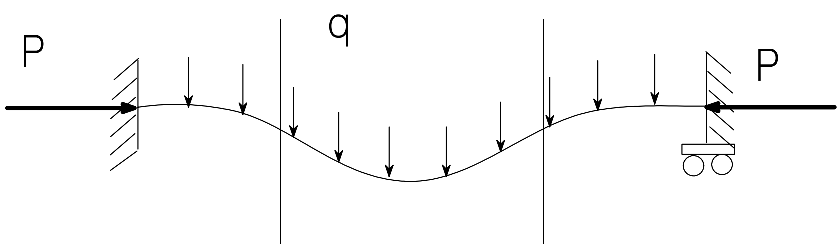
moment at ends $M_{end} := \frac{-1}{12} \cdot q \cdot L^2$

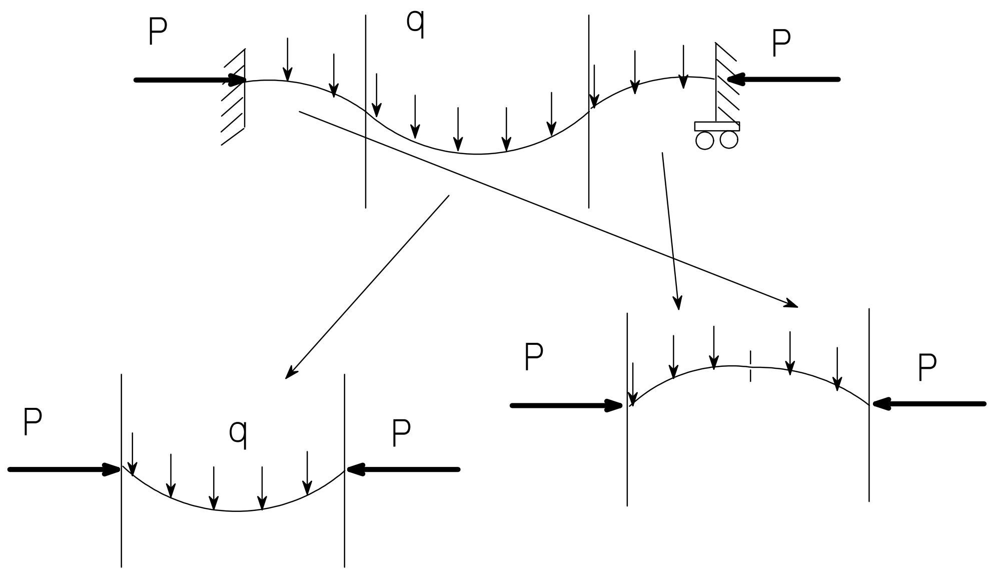
fraction of δo associated with end section $\gamma := 0.444$

deflection due to q $\delta_o := \frac{1}{384} \cdot \frac{q}{E \cdot I} \cdot L^4$



make into two "simply supported" (based on M(x) = 0) problems





<u>center</u>

$L_{cent} := \nu \cdot L$

$\delta_{cent} := (1-\gamma)\cdot\delta_o$

$M_{center} \rightarrow \frac{1}{24}\cdot q\cdot L^2$

<u>ends</u>

$L_{end} := (1-\nu)\cdot L$

$\delta_{cent} := \gamma\cdot\delta_o$

$M_{end} \rightarrow \frac{-1}{12}\cdot q\cdot L^2$

apply beam column pinned pinned relationships to each segment

$M_{max} := M_0 + P\cdot\phi\cdot(\delta_0 + \Delta)$ $\quad \sigma_{max} := \frac{P}{A} + \frac{M_{max}}{Z}$ $\quad \sigma_Y = \frac{P_{ult}}{A} + \frac{M_0}{Z} + \frac{P_{ult}\cdot(\delta_0+\Delta)}{\left(1-\frac{P_{ult}}{P_E}\right)\cdot Z}$ => $\sigma_{max} := \sigma_Y$

using $\quad \phi_1(P\_over\_Pcr) := \frac{1}{1 - P\_over\_Pcr}$

which after rearranging and defining some non-dimensional factors becomes:

$$\sigma_u(\lambda,\eta,\mu) := \left[\frac{1}{2}\cdot\left(1-\mu+\frac{1+\eta}{\lambda^2}\right) - \sqrt{\frac{1}{4}\cdot\left(1-\mu+\frac{1+\eta}{\lambda^2}\right)^2 - \frac{1-\mu}{\lambda^2}}\right]\cdot\sigma_Y$$

see beam column summary at this point