Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

Normal Modes

- Numerical Approaches
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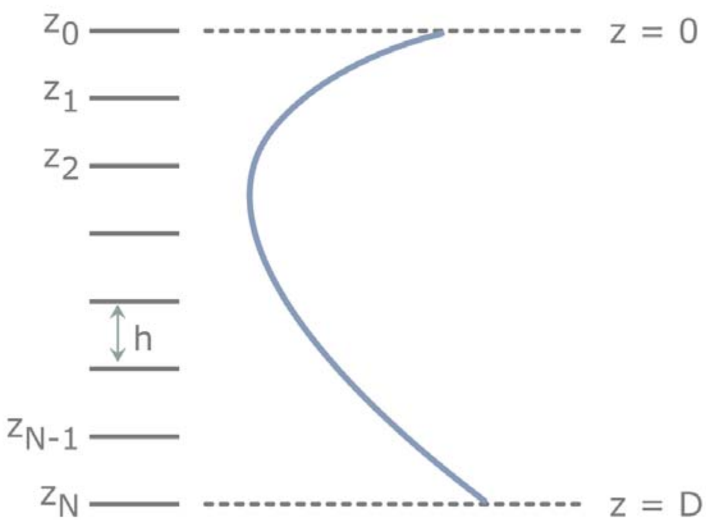
  - Group Velocity (5.8.1)

Numerical Approaches



Finite Difference Formulation

Depth-separated Helmholtz Equation - Source

$$(\mathbf{C} - k_r^2 \mathbf{I})\, \mathbf{x} = \mathbf{b} \,,$$

Modal Eigenvalue Problem

$$(\mathbf{C} - k_r^2 \mathbf{I})\, \mathbf{x} = \mathbf{0} \,.$$

Algebraic Eigenvalue Problem

$$\det A(k_r^2) = 0 \,.$$

Algebraic Eigenvalue Problem

$$\mathbf{A}(k_r^2)\,\mathbf{\Psi} = 0\,.$$

Field Vector

$$\mathbf{\Psi} = \Psi_0, \Psi_1, \ldots, \Psi_N$$

Tri-diagonal Coefficient Matrix

$$\mathbf{A} = \begin{bmatrix} d_0 & e_1 & & & & & \\ e_1 & d_1 & e_2 & & & & \\ & e_2 & d_2 & e_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & e_{N-2} & d_{N-2} & e_{N-1} & \\ & & & & e_{N-1} & d_{N-1} & e_N \\ & & & & & e_N & d_N \end{bmatrix},$$

$$\begin{aligned} d_0 &= \frac{-2 + h^2\left[\omega^2/c^2(z_0) - k_r^2\right]}{2\,h\rho} + \frac{f^T(k_r^2)}{g^T(k_r^2)}\,, \\ d_j &= \frac{-2 + h^2\left[\omega^2/c^2(z_j) - k_r^2\right]}{h\rho}\,, \qquad j = 1, \ldots N-1\,, \\ d_N &= \frac{-2 + h^2\left[\omega^2/c^2(z_N) - k_r^2\right]}{2\,h\rho} - \frac{f^B(k_r^2)}{g^B(k_r^2)}\,, \end{aligned}$$

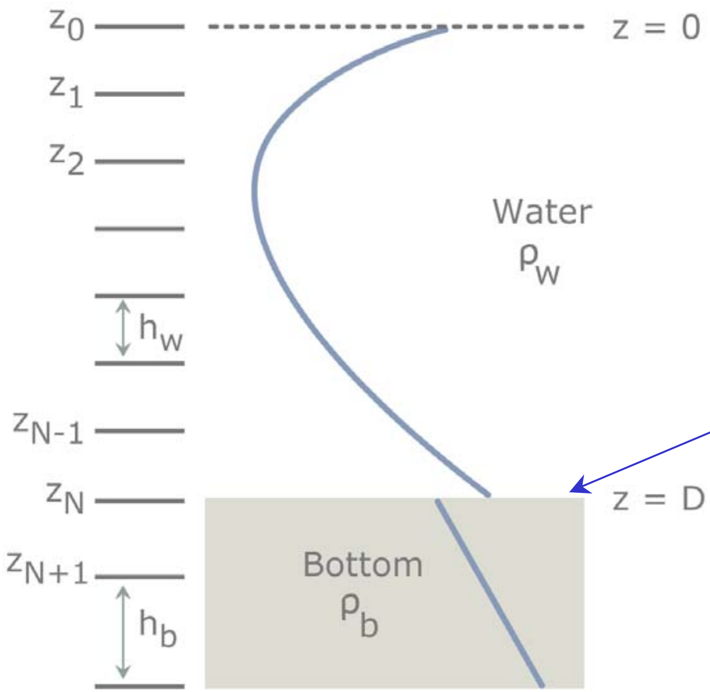
$$e_j = \frac{1}{h\rho}, \qquad j = 1, \ldots N\,.$$

Free Surface

$$f/g \to \infty \Rightarrow \Psi_0 = 0$$

Solving the Modal Eigenvalue Problem

1. QR algorithm - designed for subsets of modes.
2. Sturm's method
   - Bi-section, Sturm sequences
   - Newton's Method, Sturm sequences
   - Inverse Iteration
   - Richardson Extrapolation



Treatment of Interfaces

Water

$$\Psi_{j-1}+\left\{-2+h_w^2\left[\frac{\omega^2}{c^2(z_j)}-k_r^2\right]\right\}\Psi_j+\Psi_{j+1}=0\,,\qquad j=1,\ldots N-1\,,$$

Bottom

$$\Psi_{j-1}+\left\{-2+h_b^2\left[\frac{\omega^2}{c^2(z_j)}-k_r^2\right]\right\}\Psi_j+\Psi_{j+1}=0\,,\qquad j=N+1,\ldots$$

Interface Conditions

Continuity of Pressure

$$\Psi_N=\Psi(D^-)=\Psi(D^+)$$

Continuity of Particle Velocity

$$\frac{d\Psi(D)/dz}{\rho_w}=\frac{d\Psi(D)/dz}{\rho_b}\,,$$

$$\left\{\frac{\Psi_N-\Psi_{N-1}}{h_w}-\left[\frac{\omega^2}{c^2(D^-)}-k_r^2\right]\Psi_N\frac{h_w}{2}\right\}\Big/\rho_w$$

$$=\left\{\frac{\Psi_{N+1}-\Psi_N}{h_b}+\left[\frac{\omega^2}{c^2(D^+)}-k_r^2\right]\Psi_N\frac{h_b}{2}\right\}\Big/\rho_b\,,$$

$$\frac{\Psi_{N-1}}{h_w\rho_w}+\frac{-\Psi_N+\left[\omega^2/c^2(D^-)-k_r^2\right]\Psi_N\,h_w^2/2}{h_w\rho_w}$$

$$+\frac{-\Psi_N+\left[\omega^2/c^2(D^+)-k_r^2\right]\Psi_N\,h_b^2/2}{h_b\rho_b}+\frac{\Psi_{N+1}}{h_b\rho_b}=0\,.$$

Mode Normalization

$$N_m = \int_0^D \frac{\Psi_m^2(z)}{\rho(z)}\, dz - \frac{1}{2k_{rm}} \left.\frac{d(f/g)^T}{dk_r}\right|_{k_{rm}} \Psi_m^2(0) + \frac{1}{2k_{rm}} \left.\frac{d(f/g)^B}{dk_r}\right|_{k_{rm}} \Psi_m^2(D).$$

$$\int_0^D \frac{\Psi_m^2(z)}{\rho(z)}\, dz \simeq \frac{D}{N}\left(\frac{1}{2}\phi_0 + \phi_1 + \phi_2 + \cdots + \phi_{N-1} + \frac{1}{2}\phi_N\right) ,$$

$$\phi_j = \frac{\Psi_j^2}{\rho(z_j)} .$$

Other Methods

Layer Method

- Analytical Solution in each layer
- Direct Global Matrix as for Wavenumber Integration
- Search for zeros of determinant.
- Modal amplitude through Wronskian

Numerov's method

$$\Psi''(z) + \left[\frac{\omega^2}{c^2(z)} - k_r^2\right]\Psi(z) = 0\,,$$

$$\left(\frac{1}{h^2} + \frac{1}{12}\,k^2_{z,j-1}\right)\Psi_{j-1} + \left(-\frac{2}{h^2} + \frac{10}{12}\,k^2_{z,j}\right)\Psi_j + \left(\frac{1}{h^2} + \frac{1}{12}\,k^2_{z,j+1}\right)\Psi_{j+1} = 0,$$

$$k^2_{z,j} = \frac{\omega^2}{c^2(z_j)} - k_r^2\,.$$

- Standard scheme: $O(h^2)$
- Numorov's method: $O(h^4)$. Twice CPU time

Shooting Methods

Modal Equations

$$\frac{d^2\Psi_m}{dz^2} + \left[\frac{\omega^2}{c^2(z)} - k_{rm}^2\right]\Psi_m = 0\,,$$

$$\Psi_m(0) = 0, \quad \frac{d\Psi_m}{dz}(D) = 0$$

Initial-Value Problem

$$\frac{d\Psi_m}{dz}(0) = 1$$

Finite Difference Recursion

$$\begin{aligned}\Psi_0 &= 0\,,\\ \Psi_1 &= h\,,\end{aligned}$$

$$\Psi_{j+1} = -\Psi_{j-1} + \left\{2 - h^2\left[\frac{\omega^2}{c^2(z_j)} - k_r^2\right]\right\}\Psi_j\,, \qquad j = 1, \ldots N\,.$$

Rigid-Bottom Boundary Condition

$$\Delta(k_r^2) = \frac{\Psi_{N+1} - \Psi_{N-1}}{2h} = 0\,.$$

Numerical Stability

Munk profile: Refracted-refracted Modes

[See Fig 5.15 in Jensen, Kuperman, Porter and Schmidt. *Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.]

Evanescent Regions
Modal Equation supports both growing and decaying evanescent components

Perturbational Treatment of Loss Mechanisms

$$\rho(z)\left[\frac{1}{\rho(z)}\,\Psi_m'(z)\right]' + \left[K^2(z) - k_{rm}^2\right]\Psi_m(z) = 0\,,$$

$$\Psi_m(0) = 0\,, \qquad \frac{d\Psi_m(D)}{dz} = 0\,, \qquad \int_0^D \frac{\Psi_m^2(z)}{\rho(z)}\,dz = 1\,,$$

$$K^2(z) = \omega^2/c^2(z)$$

Medium Wavenumber Perturbation

$$K^2(z) = K_0^2(z) + \epsilon K_1^2(z) + \cdots\,,$$

Insert in Modal Equations and arrange by order

$$\Psi(z) = \Psi_0(z) + \epsilon\Psi_1(z) + \cdots\,,$$

$$k_r^2 = k_{r0}^2(z) + \epsilon k_{r1}^2 + \cdots$$

$O(1)$ Equations

$$\rho(z)\left[\frac{1}{\rho(z)}\,\Psi_0'(z)\right]' + \left[K_0^2(z) - k_{r0}^2\right]\Psi_0(z) = 0\,,$$

$$\Psi_0(0) = 0\,, \qquad \frac{d\Psi_0(D)}{dz} = 0\,, \qquad \int_0^D \frac{\Psi_0^2(z)}{\rho(z)}\,dz = 1\,.$$

Lossless eigenvalue problem that can be solved on the real axis

$O(\epsilon)$ Equations

$$\rho(z)\left[\frac{1}{\rho(z)}\,\Psi_1'(z)\right]' + \left[K_0^2(z) - k_{r0}^2\right]\Psi_1(z) = -\left[K_1^2(z) - k_{r1}^2\right]\Psi_0(z)\,,$$

$$\Psi_1(0) = 0\,, \qquad \frac{d\Psi_1(D)}{dz} = 0\,, \qquad \int_0^D \frac{\Psi_1^2(z)}{\rho(z)}\,dz = 1\,.$$

Fredholm Alternate Theorem

$$-[k_1^2(z) - k_{r1}^2]\Psi_0(z) \text{ orthogonal to } \Psi_0(z)$$

$$\int_0^D \left[K_1^2(z) - k_{r1}^2\right]\frac{\Psi_0^2(z)}{\rho(z)}\,dz = 0\,,$$

$$\Rightarrow$$

$$k_{r1}^2 = \int_0^D \frac{K_1^2(z)\,\Psi_0^2(z)}{\rho(z)}\,dz\,,$$

Procedure

1. Complex sound speed $c(z) = c_r(z) + ic_i(z)$ and complex wavenumber $K^2(z) = K_r^2(z) + iK_i^2(z) = \omega/c(z)$.

2. Use real part K_r^2 to find real eigenvalues k_r and eigenfunctions $\Psi(z)$ for unperturbed, $O(1)$ problem.

3. Denote the perturbation term by $\epsilon K_1^2 = iK_i^2(z)$ and the corresponding perturbation to the eigenvalue by $\epsilon k_{r1}^2 = i\gamma^2$.

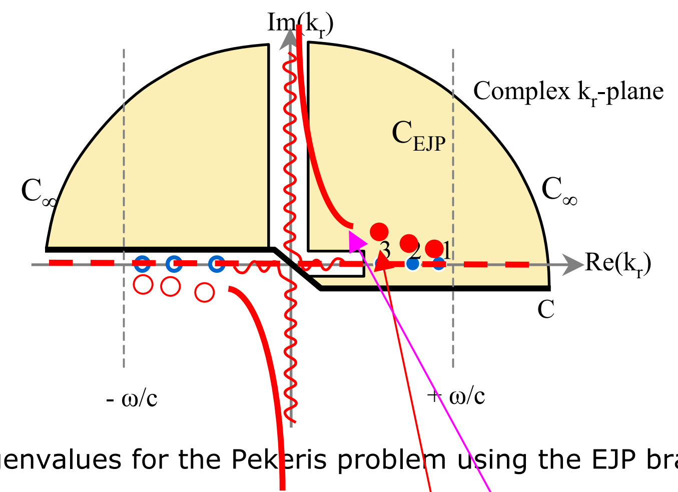
$$
\begin{aligned}
i\gamma^2 = \epsilon k_{r1}^2 &= \int_0^D \frac{\epsilon K_1^2(z)\,\Psi^2(z)}{\rho(z)}\,dz \\
&= \int_0^D \frac{iK_i^2(z)\,\Psi^2(z)}{\rho(z)}\,dz\,,
\end{aligned}
$$

Imaginary Wavenumber Perturbation

$$
\gamma^2 = \int_0^D \frac{K_i^2(z)\,\Psi^2(z)}{\rho(z)}\,dz\,.
$$

Generalized, Penetrable-Bottom Problem. (ad hoc)

$$
\gamma^2 = \int_0^\infty \frac{K_i^2(z)\,\Psi^2(z)}{\rho(z)}\,dz\,.
$$



[See Jensen, Fig 2.30]

Location of eigenvalues for the Pekeris problem using the EJP branch cut.

Contour Integral

$$\int_{-\infty}^{\infty} + \int_{C_\infty} + \int_{C_{EJP}} = 2\pi i \sum_{m=1}^{M} \mathrm{res}(k_{rm}) \; ,$$

$\mathrm{res}(k_{rm})$: residue of the mth pole enclosed by the contour.

$$p(r,z) = \frac{i}{2} \sum_{m=1}^{M} \frac{p_1(z_<; k_{rm})\, p_2(z_>; k_{rm})}{\partial W(z_s; k_r)/\partial k_r|_{k_r=k_{rm}}} H_0^{(1)}(\boxed{k_{rm}} r)\, k_{rm} - \boxed{\int_{C_{EJP}}} ,$$

Modal Group Velocity

$$u_n(\omega) = \frac{d\omega}{dk_{rn}} \, .$$

$$u_n \simeq \frac{(\omega + \Delta\omega) - \omega}{k_{rn}(\omega + \Delta\omega) - k_{rn}(\omega)} \, .$$

Perturbation Formulation

$$K^2(z) = \frac{(\omega + \Delta\omega)^2}{c^2(z)} \simeq \frac{\omega^2}{c^2(z)} + \frac{2\,\Delta\omega\,\omega}{c^2(z)} \, .$$

$$\begin{aligned} K^2 &= K_0^2 + \epsilon K_1^2 \\ K_0^2 &= \omega^2/c^2, \\ K_1^2 &= 2\omega/c^2 \\ \epsilon &= \Delta\omega \end{aligned}$$

$$k_{r1}^2 = \int_0^D \frac{2\omega}{c^2(z)} \frac{\Psi_0^2(z)}{\rho(z)} \, dz \, .$$

Finite Difference Perturbation

$$k_r^2(\omega + \Delta\omega) \simeq k_{r0}^2(\omega) + \Delta\omega \, k_{r1}^2$$

$$\frac{k_r^2(\omega + \Delta\omega) - k_{r0}^2(\omega)}{\Delta\omega} \simeq k_{r1}^2 \, .$$

$$\frac{k_r^2(\omega + \Delta\omega) - k_{r0}^2(\omega)}{\Delta\omega} \rightarrow_{\Delta\omega \rightarrow 0} \frac{dk_r^2}{d\omega} \, .$$

$$\frac{d(k_r^2)}{d\omega} = 2k_r \frac{dk_r}{d\omega} = k_{r1}^2 \, .$$

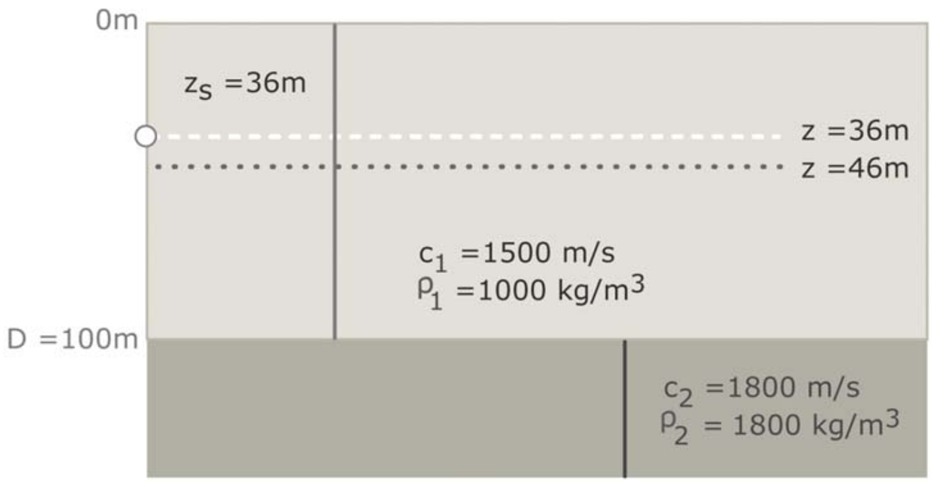
Modal Group Slowness

$$\frac{dk_r}{d\omega} = \frac{k_{r1}^2}{2k_r} = \frac{\omega}{k_r} \int_0^D \frac{\Psi_0^2(z)}{\rho(z)\, c^2(z)} \, dz \, .$$

Modal Group Speed - Penetrable Bottom

$$u_n = \frac{d\omega}{dk_{rn}} = \frac{k_{rn}}{\omega}\left[\int_0^\infty \frac{\Psi_0^2(z)}{\rho(z)\,c^2(z)}\,dz\right]^{-1}$$

Pekeris Waveguide



[See Jensen, Fig 2.28b]

Modal Group Speed - Penetrable Bottom

$$u_n = \frac{d\omega}{dk_{rn}} = \boxed{\frac{k_{rn}}{\omega}} \left[\int_0^{\infty} \frac{\Psi_0^2(z)}{\rho(z)\, c^2(z)}\, dz \right]^{-1}$$

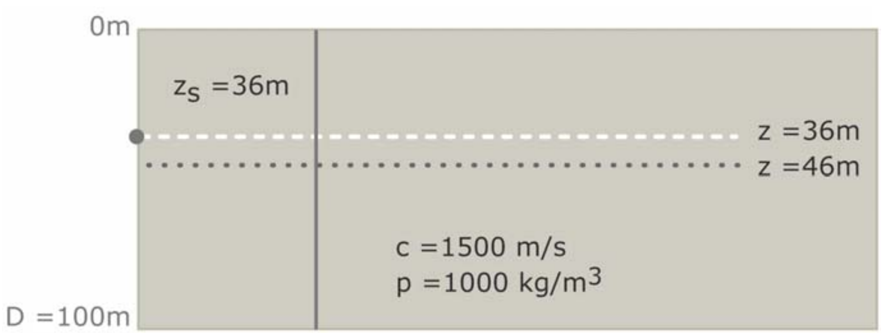
$$\frac{1}{\boxed{v_n}}$$

Isovelocity, Ideal Waveguide

Modal
Phase Velocity

$$u_n = \frac{k_{rn} c^2}{\omega} = \frac{c^2}{\boxed{v_n}}$$

Ideal Waveguide



[See Jensen, Fig 2.22]