1.138J/2.062J/18.376J, WAVE PROPAGATION

Fall, 2006 MIT

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Homework no. 2

Given Sep 26,2006. Due October 5, 2006.

In all exercises, please describe the physical meaning of your mathematical results. Use graphics if it can help the explanation. If you do any numerical computations, feel free to use Matlab.

1. Reflection from a semi-infinite rod. Consider the longitudinal waves in a semei-infinite elastic rod of uniform cross section. The end at $x = 0$ is stress-free. There is no external stress along the rod. The initial displacement and velocity are :

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x), \quad x > 0.$$

Find the deflection in the rod for all time $t > 0$ by using the method of images.

2. Read §1. Chapter one, Notes.

Consider an infinitely long string taut with tension T, $-\infty < x < \infty$ free from any lateral support. A concentrated mass M is attached to the string at the origin. Show first that Newton's law for the mass requires that

$$M\frac{\partial^2}{\partial t^2}V_0(t) = -T\frac{\partial}{\partial x}V_-(0-,t) + T\frac{\partial}{\partial x}V_+(0+,t), \quad t > 0. \tag{H.2.1}$$

where $V_0(t)$ is the displacement of the mass, V_- the string displacement on the left side ($x < 0$) and V_+ the string displacement on the right ($x > 0$).

An incident pulse with finite extent $V_I(x,t)$ arrives from $x \sim -\infty$. Its front arrives at $x = 0$ when $t = 0$, i.e., $V_I(0,0) = 0$. Find the reflected and the transmitted waves and the motion of the mass for all $t > 0$.

Suggestions:

Take as the solution:

$$V_-(x,t) = V_I\left(t - \frac{x}{c}\right) + V_R\left(t + \frac{x}{c}\right), \quad x < 0$$

$$V_+(x,t) = V_T\left(t - \frac{x}{c}\right), \quad x > 0.$$

here the subscripts mean: I= incident, R= reflected and T= transmitted. From the boundary condition at $x = 0$ find a differential equation for $V_0(t)$. State proper initial conditions and solve for $V_0(t)$, hence get $V_R\left(t + \frac{x}{c}\right)$ and $V_T\left(t - \frac{x}{c}\right)$.

To see the physics more explicitly , you may specify the pulse, e.g., half of a sine curve and carry out the necessary integration.

3. Two semi-infinite cylindrical rods of different materials but the same uniform cross section S are butted toegether at $x = 0$. The elastic constant is E_1 in $x < 0$ and E_2 in $x > 0$. At $t = 0$ the rod on the left has a nonuniform displacement but no velocity

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x < 0$$

where $f(x)$ is nonzero only in a finite domain. The rod on the right is free of initial deformation and velocity

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x > 0$$

Find the displacement in both rods for all $t > 0$. Note that in the left rod there will be a left-going (reflected) wave after some time. In the right rod there is only a right-going wave for all time (The radiation condition).

4. **Long wave dispersion in shallow water** It can be shown by a more accurate analysis that infinitesimal long waves in shallow water are governied by the following conservation equations:

$$\zeta_t + hu_x = 0 \tag{H.2.2}$$

$$u_t + g\zeta_x - \frac{h^2}{3}u_{xxt} = 0 \tag{H.2.3}$$

1. Eliminate u by cross differentiation to get a single PDE for $\zeta(x,t)$.

2. For a sinusoidal wave

$$\zeta = \Re(Ae^{i(kx-\omega t)}) \tag{H.2.4}$$

Find the dispersion relation and examine the dependence of phase velocity and group velocity on the wavenumber.

3. Let the initial disturbance be

$$\zeta(x,0) = f(x), \quad u(x,t) = 0 \tag{H.2.5}$$

where $f(x) \neq 0$ only in a bounded region. Find $\zeta(x,t)$ by Fourier transform.

4. Study the wave disperison for large t, and describe the physics of your results.