Goals for today

– From ODE to Laplace transform

– Laplace transform definition and properties

– Using the Laplace transform to derive the flywheel response

From ODE to Laplace transform /1

Recall the ODE we derived for the car suspension:

$$m\ddot{x} + b\dot{x} + kx = b\dot{u} + ku.$$

We need to solve in two steps: homogeneous (with the right–hand side equal to zero, *i.e.* no input) and the particular solution for the given input.

The homogeneous solution to an LTI ODE is easy, because it is an exponential function. We can denote the exponent as s, then we obtain

$$m\ddot{x}_{\mathrm{h}} + b\dot{x}_{\mathrm{h}} + kx_{\mathrm{h}} = 0, \qquad x_{\mathrm{h}}(t) = A\mathrm{e}^{st}.$$

Substituting yields $\left(ms^2 + bs + k\right) A\mathrm{e}^{st} = 0$ and, since we require a non–trivial solution $(A \neq 0)$ we must have

$$ms^2 + bs + k = 0.$$

This is called the system's *characteristic equation.*
We will say more about its solutions later.

From ODE to Laplace transform /2

The particular solution is, generally, more difficult to find, because we need to *guess* its form, depending on the input function and its derivatives, as they appear on the right–hand side. However, if the input happens to be an exponential function

$$u(t) = U_0 \mathrm{e}^{s_0 t},$$

then we can easily guess that the particular solution (output) would be of the same form, *i.e.*

$$x_{\mathrm{p}}(t) = H_0 U_0 \mathrm{e}^{s_0 t}$$

Substituting into the ODE we obtain

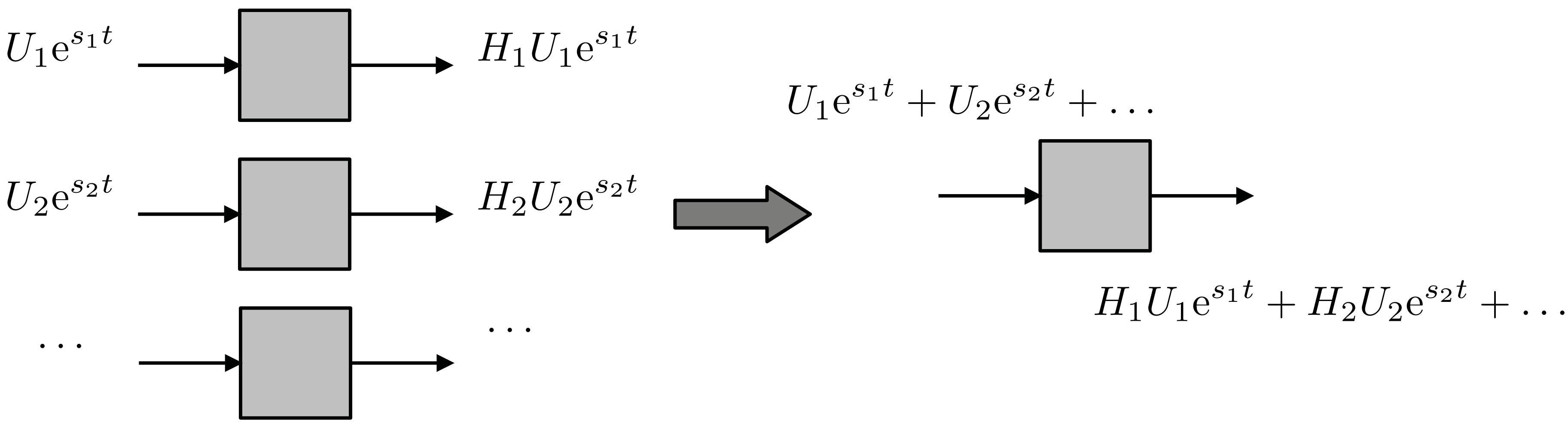
$$ms_0^2 + bs_0 + k \;\; H_0 U_0 \mathrm{e}^{s_0 t} = (bs_0 + k)\, U_0 \mathrm{e}^{s_0 t} \Rightarrow H_0 = \frac{bs_0 + k}{ms_0^2 + bs_0 + k}.$$

With this choice of H_0, the ODE is satisfied by the particular solution

$$x_{\mathrm{p}}(t) = U_0 \frac{bs_0 + k}{ms_0^2 + bs_0 + k} \mathrm{e}^{s_0 t}.$$

From ODE to Laplace transform /3

It was the insight of two French mathematicians, Fourier and Laplace (who were vicious competitors and accused each other of stealing each other's idea) that the particular solution for an exponential input can be exploited in the case of LTI (linear and time invariant) ODE as follows:



This is the *superposition principle.* Thus, if we can express any input function as a superposition of exponentials, then we can find the output of the system, also as a superposition of exponentials. This led to the idea of the Laplace and Fourier transforms. In this class, we consider the former only, as it is more appropriate for calculating *transient response, i.e.* for $t > 0$ assuming initial conditions are given for $t = 0$.

Laplace transform: definition

Given a function $f(t)$ in the time domain we define its Laplace transform $F(s)$ as

$$F(s) = \int_{0-}^{+\infty} f(t)\mathrm{e}^{-st}\mathrm{d}t.$$

We say that $F(s)$ is the frequency–domain representation of $f(t)$.

The frequency variable s is a complex number:

$$s = \sigma + j\omega,$$

where σ, ω are real numbers with units of frequency ($i.e.$ $\sec^{-1} \equiv$Hz).

We will investigate the physical meaning of σ, ω later when we see examples of Laplace transforms of functions corresponding to physical systems.

Example 1: Laplace transform of the step function

Consider the step function (*aka* Heaviside function)

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

According to the Laplace transform definition,

$$\begin{aligned} U(s) &= \int_{0-}^{+\infty} u(t)\mathrm{e}^{-st}\mathrm{d}t = \int_{0-}^{+\infty} 1 \cdot \mathrm{e}^{-st}\mathrm{d}t = \\ &= \left(\frac{1}{-s}\mathrm{e}^{-st}\right)\Bigg|_{0-}^{+\infty} = \frac{1}{-s}\Big(0 - 1\Big) = \\ &= \frac{1}{s}. \end{aligned}$$

Interlude: complex numbers: what does 1/*s* mean?

Recall that $s = \sigma + j\omega$. The real variables σ, ω (both in frequency units) are the real and imaginary parts, respectively, of s. (We denote $j^2 = -1$.)

Therefore, we can write

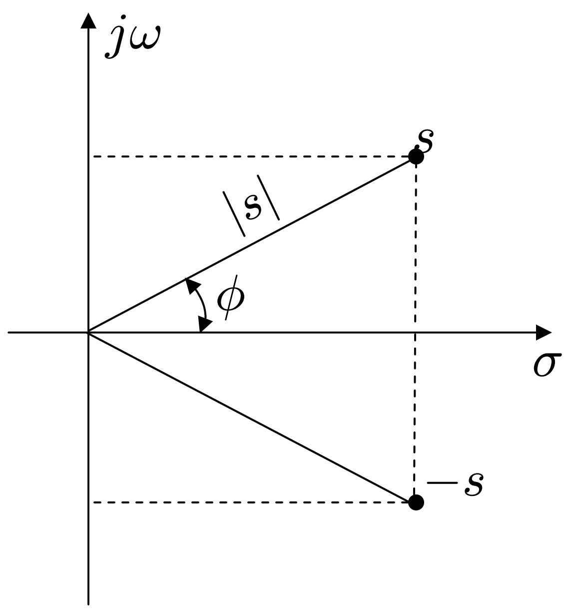
$$\frac{1}{s} = \frac{1}{\sigma + j\omega} = \frac{\sigma - j\omega}{(\sigma + j\omega)(\sigma - j\omega)} = \frac{\sigma - j\omega}{\sigma^2 + \omega^2}.$$

Alternatively, we can represent the complex number s in polar form $s = |s|\, \mathrm{e}^{j\phi}$,

where $|s| = \left(\sigma^2 + \omega^2\right)^{1/2}$ is the magnitude and $\phi \quad \angle s = \mathrm{atan}\,(\omega/\sigma)$ the phase of s.

It is straightforward to derive

$$\frac{1}{s} = \frac{1}{|s|}\mathrm{e}^{-j\phi} \Rightarrow \left|\frac{1}{s}\right| = \frac{1}{|s|} \qquad \text{and} \qquad \angle\frac{1}{s} = -\angle s.$$



Example 2: Laplace transform of the exponential

Consider the decaying exponential function beginning at $t = 0$

$$f(t) = \mathrm{e}^{-at}u(t),$$

where $a > 0$ (note the presence of the step function in the above formula.)

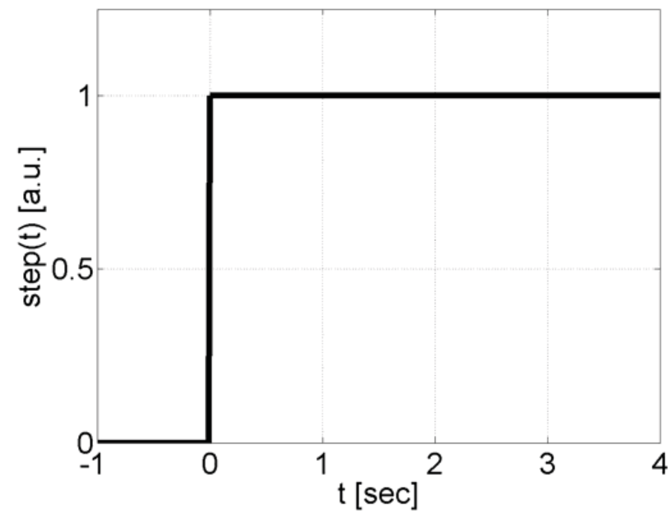
Again we apply the Laplace transform definition,

$$\begin{aligned} F(s) &= \int_{0-}^{+\infty} \mathrm{e}^{-at}u(t)\mathrm{e}^{-st}\mathrm{d}t = \int_{0-}^{+\infty} \mathrm{e}^{-(s+a)t}\mathrm{d}t = \\ &= \left(\frac{1}{-(s+a)}\mathrm{e}^{-(s+a)t}\right)\Bigg|_{0-}^{+\infty} = \frac{1}{-(s+a)}\Big(0-1\Big) = \\ &= \frac{1}{s+a}. \end{aligned}$$

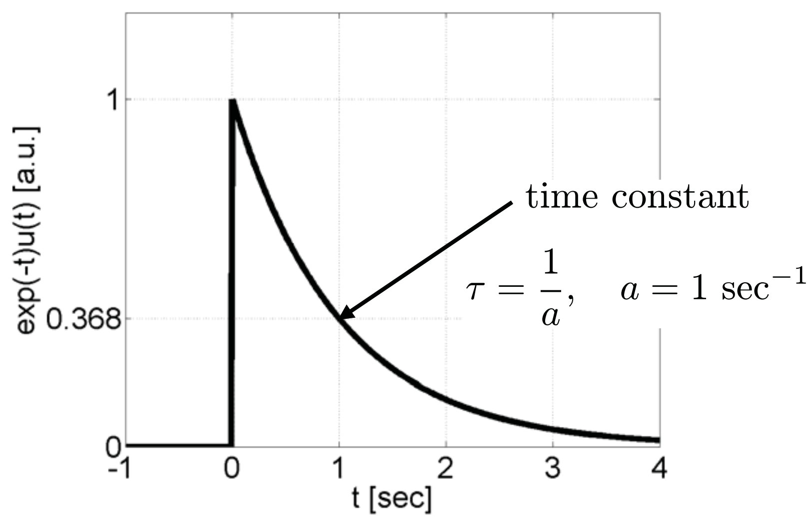
Laplace transforms of commonly used functions

f(t)	*F(s)*
$\delta(t)$	1
$\boxed{u(t)}$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$\boxed{e^{-at}u(t)}$	$\frac{1}{s+a}$
$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Nise Table 2.1



Step function (*aka* Heaviside)



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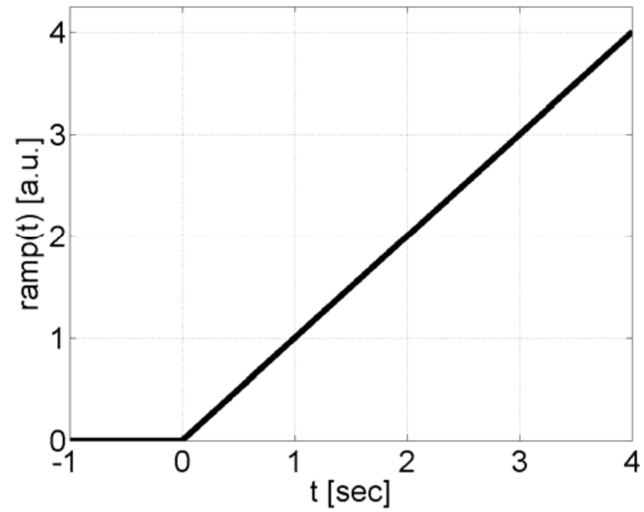
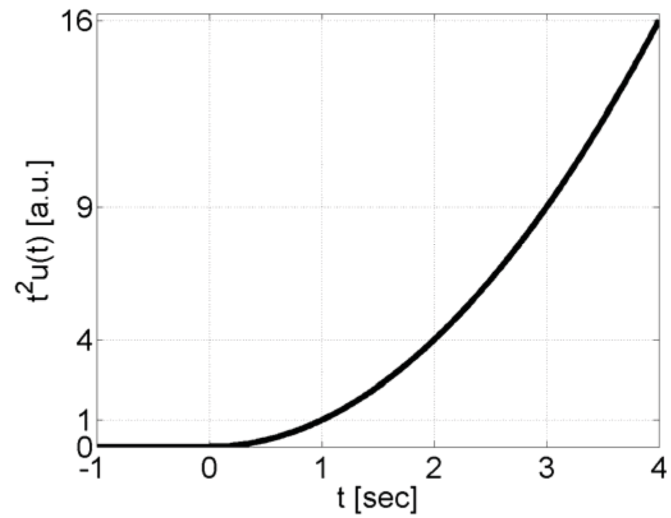
Nise Table 2.1

Polynomials



Ramp function



Quadratic function

$n = 2$

Laplace transforms of commonly used functions

Sinusoids

f(t)	*F(s)*
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\boxed{\sin \omega t u(t)}$	$\frac{\omega}{s^2+\omega^2}$
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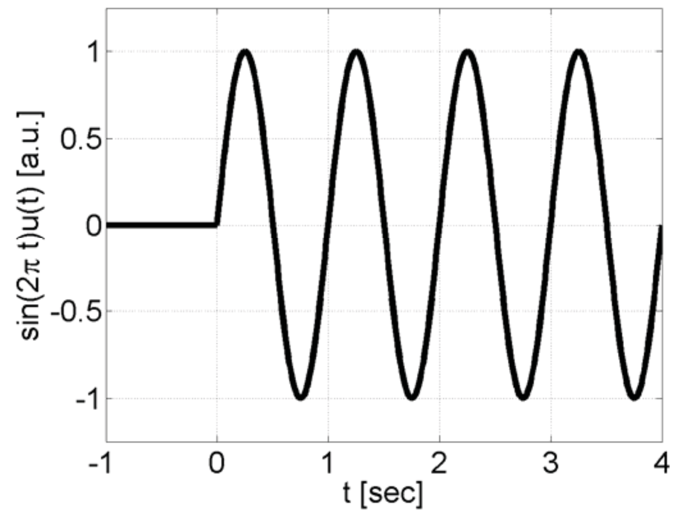
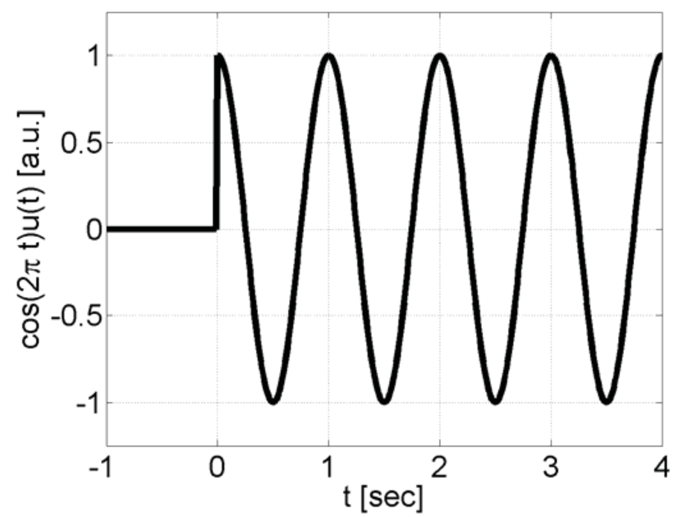
Nise Table 2.1





Laplace transforms of commonly used functions

f(t)	F(s)
$\boxed{\delta(t)}$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
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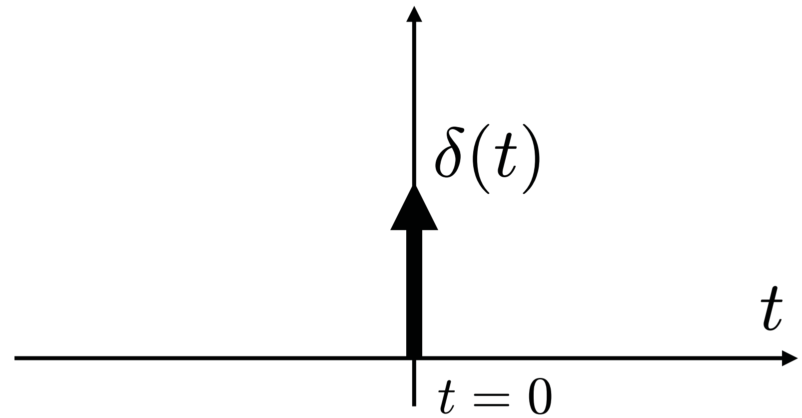
Nise Table 2.1

Impulse function (*aka* Dirac function)



It represents a pulse of

- <u>infinitessimally small</u> duration; and
- <u>finite</u> energy.

Mathematically, it is defined by the properties

$$\int_{-\infty}^{+\infty} \delta(t) = 1; \qquad \text{(unit energy) and}$$

$$\int_{-\infty}^{+\infty} \delta(t) f(t) = f(0) \qquad \text{(sifting.)}$$

Properties of the Laplace transform

Let $F(s)$, $F_1(s)$, $F_2(s)$ denote the Laplace transforms of $f(t)$, $f_1(t)$, $f_2(t)$, respectively. We denote $\mathcal{L}\left[f(t)\right] = F(s)$, etc.

- Linearity

$\mathcal{L}\left[K_1 f_1(t) + K_2 f_2(t)\right] = K_1 F_1(s) + K_2 F_2(s)$,
where K_1, K_2 are complex constants.

- Differentiation

The differentiation property is the one that we'll find most useful in solving linear ODEs with constant coeffs.

  - $\mathcal{L}\left[\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right] = sF(s) - f(0-)$;
  - $\mathcal{L}\left[\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\right] = s^2 F(s) - sf(0-) - \dot{f}(0)$; and
  - $\mathcal{L}\left[\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0-)$.

- Integration

$$\mathcal{L}\left[\int_{0-}^{t} f(\xi)\mathrm{d}\xi\right] = \frac{F(s)}{s}.$$

A more complete set of Laplace transform properties is in Nise Table 2.2.
We'll learn most of these properties in later lectures.

Inverting the Laplace transform

Consider

$$F(s) = \frac{2}{(s+3)(s+5)}. \tag{1}$$

We seek the inverse Laplace transform $f(t) = \mathcal{L}^{-1}[F(s)]$:*i.e.*, a function $f(t)$ such that $\mathcal{L}[f(t)] = F(s)$.

Let us attempt to re–write $F(s)$ as

$$F(s) = \frac{2}{(s+3)(s+5)} = \frac{K_1}{s+3} + \frac{K_2}{s+5}. \tag{2}$$

That would be convenient because we know the inverse Laplace transform of the $1/(s+a)$ function (it's a decaying exponential) and we can also use the linearity theorem to finally find $f(t)$. All that'd be left to do would be to find the coefficients K_1, K_2.

This is done as follows: first multiply both sides of (2) by $(s+3)$. We find

$$\frac{2}{s+5} = K_1 + \frac{K_2(s+3)}{s+5} \overset{s=-3}{\Longrightarrow} K_1 = \frac{2}{-3+5} = 1.$$

Similarly, we find $K_2 = -1$.

Inverting the Laplace transform

So we have found

$$F(s) = \frac{2}{(s+3)(s+5)} = \frac{1}{s+3} - \frac{1}{s+5}.$$

From the table of Laplace transforms (Nise Table 2.1) we know that

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] &= \mathrm{e}^{-3t}u(t) \qquad \text{and} \\ \mathcal{L}^{-1}\left[\frac{1}{s+5}\right] &= \mathrm{e}^{-5t}u(t). \end{aligned}$$

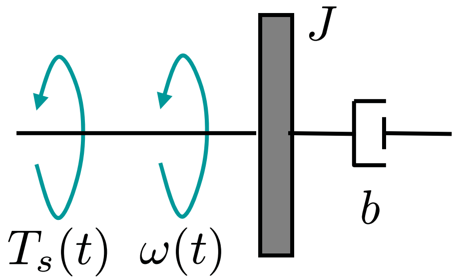
Using these and the linearity theorem we obtain

$$\mathcal{L}^{-1}\left[F(s)\right] = \mathcal{L}^{-1}\left[\frac{2}{(s+3)(s+5)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+3} - \frac{1}{s+5}\right] = \mathrm{e}^{-3t} - \mathrm{e}^{-5t}.$$

The process we just followed is known as **partial fraction expansion.**

Use of the Laplace transform to solve ODEs

- Example: motor-shaft system from Lecture 2 (and labs)



$$J\dot{\omega}(t) + b\omega(t) = T_s(t),$$

where $T_s(t) = T_0 u(t)$ (step function)

and $\omega(t=0) = 0$ (no spin–down).

Taking the Laplace transform of both sides,

$$Js\Omega(s) + b\Omega(s) = \frac{T_0}{s} \Rightarrow \Omega(s) = \frac{T_0}{b}\frac{1}{s\Big((J/b)s+1\Big)} = \frac{T_0}{b}\frac{1}{s(\tau s+1)},$$

where $\tau \equiv J/b$ is the time constant (see also Lecture 2).

We can now apply the partial fraction expansion method to obtain

$$\Omega(s) = \frac{T_0}{b}\left(\frac{K_1}{s} + \frac{K_2}{s\quad +1}\right) = \frac{T_0}{b}\left(\frac{1}{s} - \frac{}{s\quad +1}\right) = \frac{T_0}{b}\left(\frac{1}{s} - \frac{1}{s+(1/\tau)}\right).$$

Use of the Laplace transform to solve ODEs

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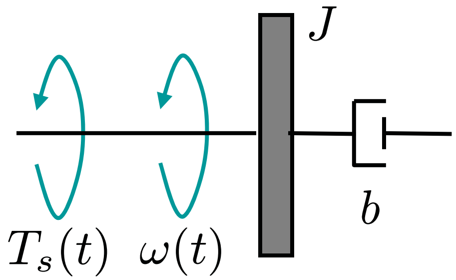

$$J\dot{\omega}(t) + b\omega(t) = T_s(t),$$

where $T_s(t) = T_0 u(t)$ (step function)

and $\omega(t=0) = 0$ (no spin–down).

We have found

$$\Omega(s) = \frac{T_0}{b}\left(\frac{1}{s} - \frac{1}{s + (1/\tau)}\right).$$

Using the linearity property and the table of Laplace transforms we obtain

$$\omega(t) = \mathcal{L}^{-1}\left[\Omega(s)\right] = \frac{T_0}{b}\left(1 - \mathrm{e}^{-t/\tau}\right),$$

in agreement with the time–domain solution of Lecture 2.