2.032 Dynamics

Fall 2004

Quiz No. 1

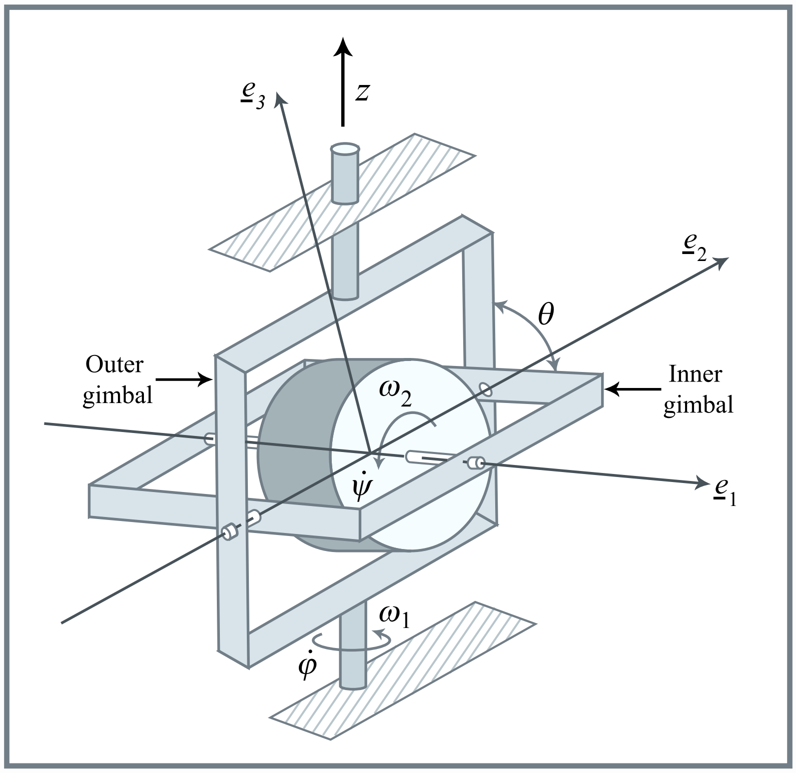
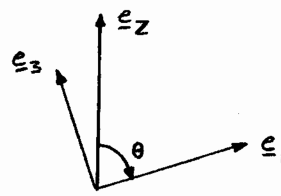


Figure by OCW.

Problem 1

123 coordinate system is fixed to the inner gimbal.



$$\underline{\omega}\big|_{123} = \omega_1 \underline{e}_z + \dot{\theta}\,\underline{e}_2$$

$$\underline{e}_z = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_3$$

$$\underline{\omega}\big|_{123} = \omega_1 \cos\theta\, \underline{e}_1 + \dot{\theta}\,\underline{e}_2 + \omega_1 \sin\theta\, \underline{e}_3$$

$$\underline{\omega}_{flywheel} = \omega_1 \underline{e}_z + \dot{\theta}\,\underline{e}_2 + \omega_2 \underline{e}_1$$

$$= (\omega_1 \cos\theta + \omega_2)\,\underline{e}_1 + \dot{\theta}\,\underline{e}_2 + \omega_1 \sin\theta\, \underline{e}_3$$

Angular momentum about point C:

$$\underline{M}_C = \dot{\underline{H}}_C \qquad (\underline{v}_C = \underline{0})$$

$$\underline{H}_C = \underline{\underline{I}}_C\, \underline{\omega}_{flywheel} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix} \begin{Bmatrix} \omega_1 \cos\theta + \omega_2 \\ \dot{\theta} \\ \omega_1 \sin\theta \end{Bmatrix} = I_1 (\omega_1 \cos\theta + \overbrace{\omega_2}^{\text{Const.}})\,\underline{e}_1 + I_2 \dot{\theta}\,\underline{e}_2 + I_2 \omega_1 \sin\theta\, \underline{e}_3$$

$$\dot{\underline{H}}_C = \overset{\circ}{\underline{H}}_C + \underline{\omega}\big|_{123} \times \underline{H}_C$$

$$\dot{\underline{H}}_C = I_1 (\dot{\omega}_1 \cos\theta - \omega_1 \dot{\theta} \sin\theta)\,\underline{e}_1 + I_2 \ddot{\theta}\,\underline{e}_2 + I_2 (\dot{\omega}_1 \sin\theta + \omega_1 \dot{\theta} \cos\theta)\,\underline{e}_3$$
$$+ (-I_2 \omega_1^2 \cos\theta \sin\theta + I_1 \omega_1^2 \sin\theta \cos\theta + I_1 \omega_1 \omega_2 \sin\theta)\,\underline{e}_2 + \left[I_2 \dot{\theta} \omega_1 \cos\theta - I_1 \dot{\theta} (\omega_1 \cos\theta + \omega_2)\right]\underline{e}_3$$

$$\underline{M}_C = M_1 \underline{e}_1 + M_3 \underline{e}_3$$

Problem 1

$\underline{M}_C = \dot{\underline{H}}_C \quad \Longrightarrow$

(a)

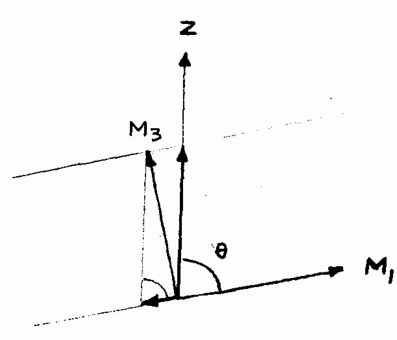
$(\underline{e}_2)$: $$\underline{I_2 \ddot{\theta} + (I_1 - I_2)\,\omega_1^2 \sin\theta \cos\theta + I_1 \omega_1 \omega_2 \sin\theta = 0}$$ equation of motion

Note that the system has one degree of freedom θ.

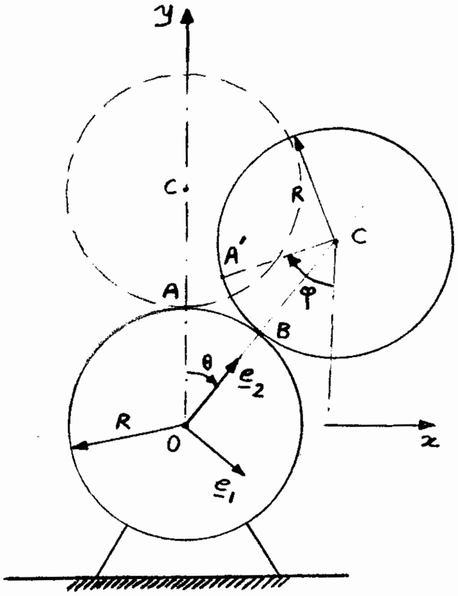
(b)

$$\left.\begin{aligned} (\underline{e}_1):\quad & M_1 = I_1\left(\dot{\omega}_1 \cos\theta - \omega_1 \dot{\theta} \sin\theta\right) \\ (\underline{e}_3):\quad & M_3 = I_2\left(\dot{\omega}_1 \sin\theta + 2\omega_1 \dot{\theta} \cos\theta\right) - I_1 \dot{\theta}\left(\omega_1 \cos\theta + \omega_2\right) \end{aligned}\right\}$$ external torques required to maintain the motion

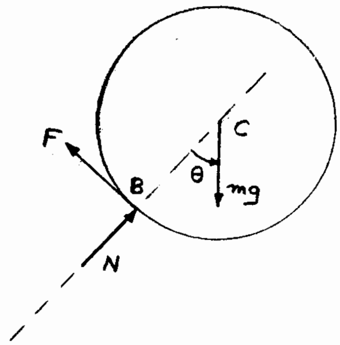
$$\begin{cases} M_Z = \dfrac{M_3}{\sin\theta} \\ M_1' = M_1 - M_3 \cot\theta \end{cases}$$



Problem 2



FBD



$\omega_{cyl.} = -\dot{\varphi}\, \underline{e}_3$

$\# DOF = 3 - 2 = 1$

123 Coordinate system rotates with $-\dot{\theta}$ about z axis as shown in the figure.

$$\underline{r}_C = 2R\,\underline{e}_2$$

$$\underline{v}_C = \underline{\dot{r}}_C = \underline{\omega}\big|_{123} \times \underline{r}_C = (-\dot{\theta}\,\underline{e}_3) \times 2R\,\underline{e}_2 = 2R\dot{\theta}\,\underline{e}_1$$

$$\underline{a}_C = \underline{\dot{v}}_C = \underline{\overset{\circ}{v}}_C + \underline{\omega}\big|_{123} \times \underline{v}_C = 2R\ddot{\theta}\,\underline{e}_1 + (-\dot{\theta}\,\underline{e}_3) \times 2R\dot{\theta}\,\underline{e}_1 = 2R\ddot{\theta}\,\underline{e}_1 - 2R\dot{\theta}^2\,\underline{e}_2$$

$\underline{v}_B = \underline{0}$ $\qquad \underline{v}_B = \underline{v}_C + \omega_{cyl.} \times \underline{r}_{CB} \longrightarrow \underline{0} = 2R\dot{\theta}\,\underline{e}_1 + (-\dot{\varphi}\,\underline{e}_3) \times (-R\,\underline{e}_2)$

$\longrightarrow \quad 2R\dot{\theta} - R\dot{\varphi} = 0 \quad \longrightarrow \quad \dot{\varphi} = 2\dot{\theta}$ (This can also be found using geometry $\varphi = 2\theta$)

Linear momentum:

$$\underline{F} = m\underline{a}_C$$

$$(mg\sin\theta - F)\,\underline{e}_1 + (N - mg\cos\theta)\,\underline{e}_2 = 2mR\ddot{\theta}\,\underline{e}_1 - 2mR\dot{\theta}^2\,\underline{e}_2 \longrightarrow \begin{cases} F = m(g\sin\theta - 2R\ddot{\theta}) \\ N = m(g\cos\theta - 2R\dot{\theta}^2) \end{cases} \quad (1)$$

Angular momentum about C:

$$\underline{M}_C = \underline{\dot{H}}_C$$

$$\underline{H}_C = \underline{\underline{I}}_C\,\underline{\omega}_{cyl.} = \left(\tfrac{1}{2}mR^2\right)(-2\dot{\theta}\,\underline{e}_3) = -mR^2\dot{\theta}\,\underline{e}_3$$

$$\underline{\dot{H}}_C = -mR^2\ddot{\theta}\,\underline{e}_3$$

Problem 2

$$\underline{M}_C = -FR\,\underline{e}_3$$

$$\therefore \quad -FR\,\underline{e}_3 = -mR^2\ddot{\theta}\,\underline{e}_3 \longrightarrow \ddot{\theta} = \frac{FR}{mR^2} = \frac{F}{mR} \qquad (2)$$

During rolling,

reaction forces N and F do not do work since $\underline{v}_B = \underline{0}$. So the only force that deos work is gravity which is potential. $\longrightarrow$ The system is conservative during rolling.

$T + V = \text{Const.}$

$$T = \frac{1}{2}m|\underline{v}_c|^2 + \frac{1}{2}\underline{\omega}^T\underline{\underline{I}}_c\,\underline{\omega} = \frac{1}{2}m(2R\dot{\theta})^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)(2\dot{\theta})^2 = 3mR^2\dot{\theta}^2$$

$$V = mgy_c = mg(2R\cos\theta) = 2mgR\cos\theta$$

$$T+V = 3mR^2\dot{\theta}^2 + 2mgR\cos\theta = E_0 = (T+V)_{\theta=0} = 2mgR \qquad (\dot{\theta}=0 \text{ at } \theta=0)$$

$$\longrightarrow \quad \dot{\theta}^2 = \frac{2g}{3R}(1-\cos\theta) \qquad (3)$$

$$(1),(2),(3) \Longrightarrow \begin{cases} F = m\left(g\sin\theta - 2\frac{F}{m}\right) \longrightarrow F = \dfrac{mg\sin\theta}{3} \\ N = m\left(g\cos\theta - \frac{4g}{3}(1-\cos\theta)\right) \longrightarrow N = mg\left(\frac{7}{3}\cos\theta - \frac{4}{3}\right) \end{cases}$$

The instant slipping begins : $F = \mu N \Longrightarrow \dfrac{mg\sin\theta}{3} = \mu mg\left(\frac{7}{3}\cos\theta - \frac{4}{3}\right)$

$$\longrightarrow \quad \underline{7\mu\cos\theta - \sin\theta = 4\mu}$$

To find θ, let $\underline{\alpha = \tan^{-1}(7\mu)} \longrightarrow 7\mu = \tan\alpha \longrightarrow \dfrac{\sin\alpha}{\cos\alpha}\cos\theta - \sin\theta = \dfrac{4}{7}\tan\alpha$

$$\longrightarrow \sin(\alpha - \theta) = \frac{4}{7}\sin\alpha \longrightarrow \underline{\theta = \alpha - \sin^{-1}\left(\frac{4}{7}\sin\alpha\right)}$$

solution is independent of the mass and radius of the cylinder.