Feedback Control System Design
2.017 Fall 2009

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10/29/2009

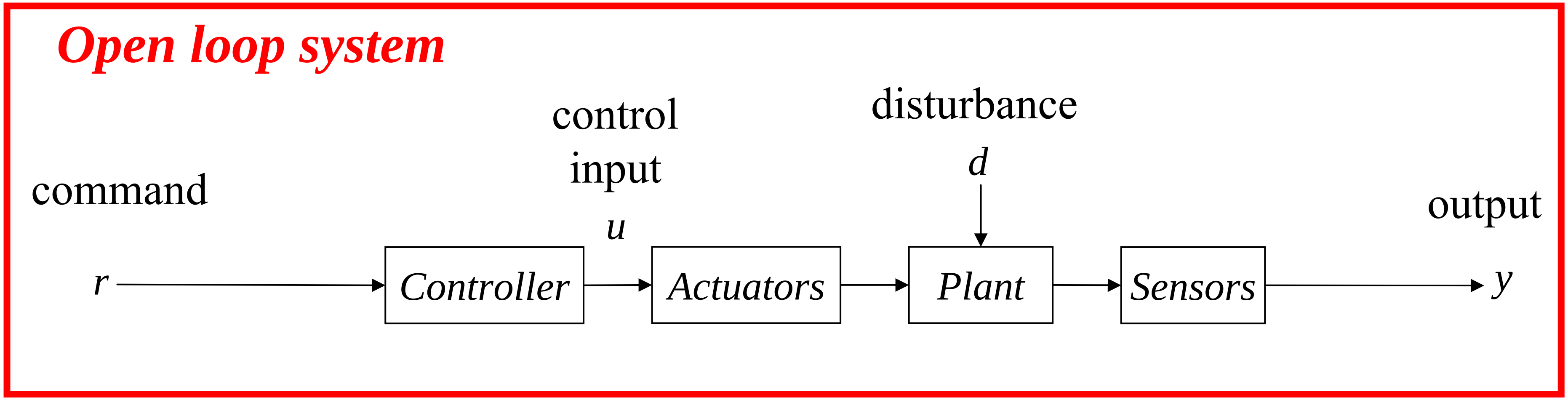
Announcements

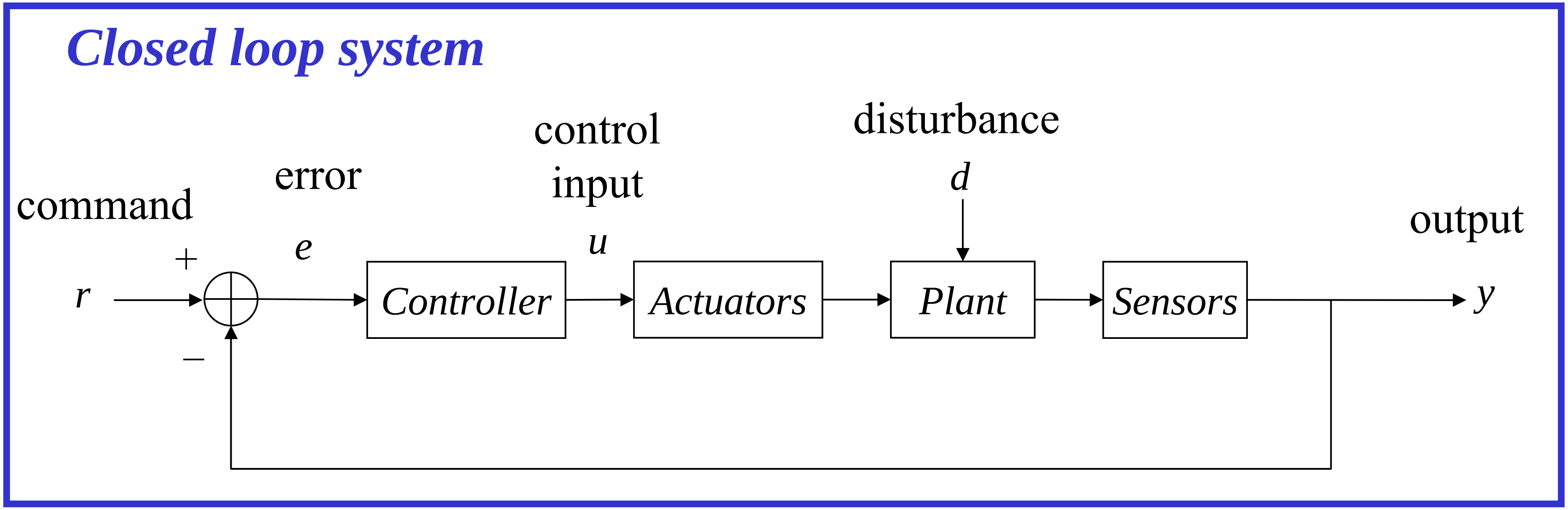
- Milestone Presentations on Nov 5 in class
  - This is 15% of your total grade:
    - 5% group grade
    - 10% individual grade
  - Email your team's PowerPoint file to Franz and Harrison by 10 am on Nov 5
  - Each team gets 30 minutes of presentation + 10 minutes of Q&A
  - Select or design your own presentation template and style

Control Systems

- An integral part of any industrial society
- Many applications including transportation, automation, manufacturing, home appliances,…
- Helped exploration of the oceans and space
- Examples:
  - Temperature control
  - Flight control
  - Process control
  - …

Types of Control Systems

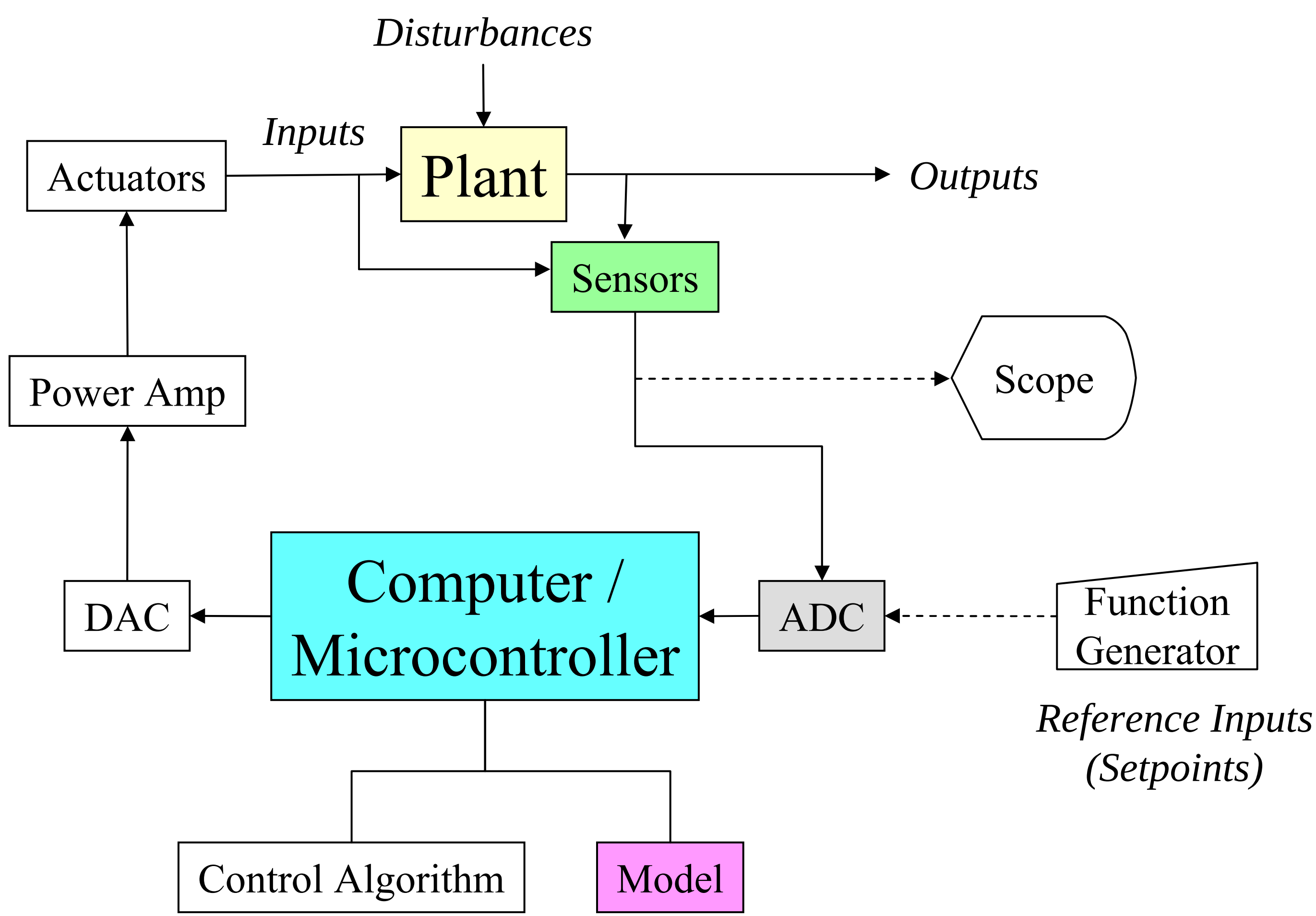




Control System Comparison

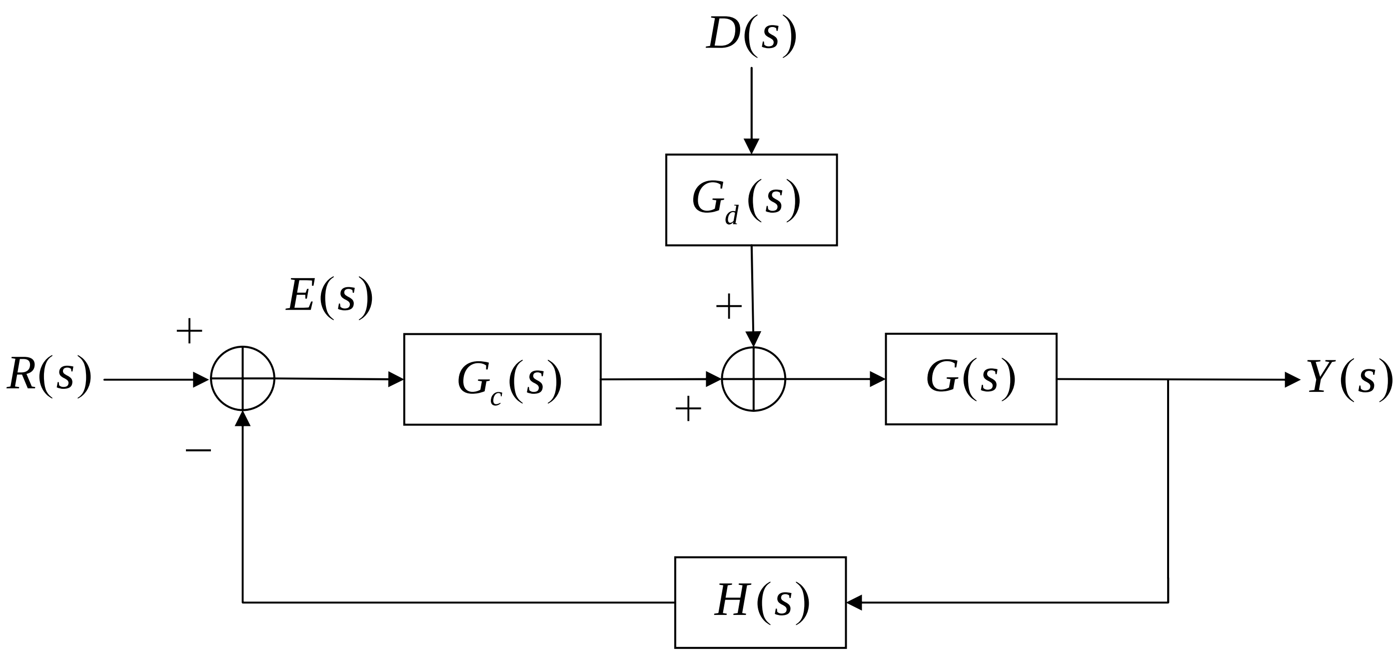
- Open loop:
  - The output variables do not affect the input variables
  - The system will follow the desired reference commands if no unpredictable effects occur
  - It can compensate for disturbances that are taken into account
  - It does not change the system stability
- Closed loop:
  - The output variables do affect the input variables in order to maintain a desired system behavior
  - Requires measurement (controlled variables or other variables)
  - Requires control errors computed as the difference between the controlled variable and the reference command
  - Computes control inputs based on the control errors such that the control error is minimized
  - Able to reject the effect of disturbances
  - Can make the system unstable, where the controlled variables grow without bound

Overview of Closed Loop Control Systems



Control System Representations

- Transfer functions (Laplace) $\dfrac{\Omega(s)}{V(s)} = \dfrac{(K_t^{-1})}{(R_m \cdot J_m / K_t^2)\cdot s + 1}$
- State-space equations (System matrices) $\begin{cases} \dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) \end{cases}$
- Block diagrams



Laplace Transform

- Convert functions of time into functions that are algebraic in the complex variables.
- Replaces differentiation & integral operations by algebraic operations all involving the complex variable.
- Allows the use of graphical methods to predict system performance without solving the differential equations of the system. These include response, steady state behavior, and transient behavior.
- Mainly used in control system analysis and design.

Laplace vs. Fourier Transform

- Laplace transform:

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt \qquad\qquad f'(t) \Rightarrow sF(s)$$

- Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

- Laplace transforms often depend on the initial value of the function
- Fourier transforms are independent of the initial value.
- The transforms are only the same if the function is the same both sides of the *y*-axis (so the unit step function is different).

System Modeling (1^{st} Order System)

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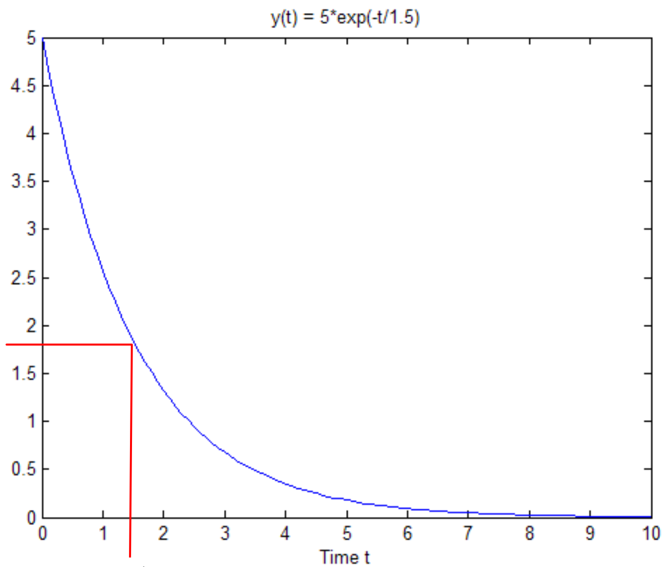
Differential equation:

$$m\dot{v}(t) + bv(t) = f(t) \xrightarrow{\text{Laplace transform}} mV(s)s + bV(s) = F(s)$$

$$m\dot{v}(t) + bv(t) = 0$$

$$v(t) = v_0(t)e^{-\left(\frac{b}{m}\right)t}$$



$5 \times e^{-1} = 1.84$

$\tau \approx 1.5$

Transfer function:

$$mV(s)s + bV(s) = F(s)$$

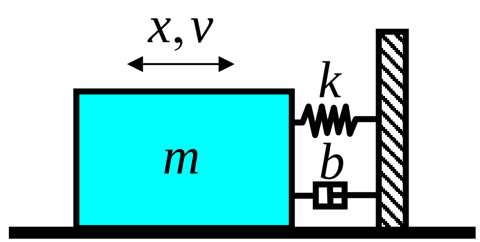
$$\frac{V(s)}{F(s)} = \frac{1}{ms+b} = \frac{(1/b)}{(m/b)s+1}$$

$$s = \frac{-1}{(m/b)} = \frac{-b}{m} = \frac{-1}{\tau}$$

Time constant

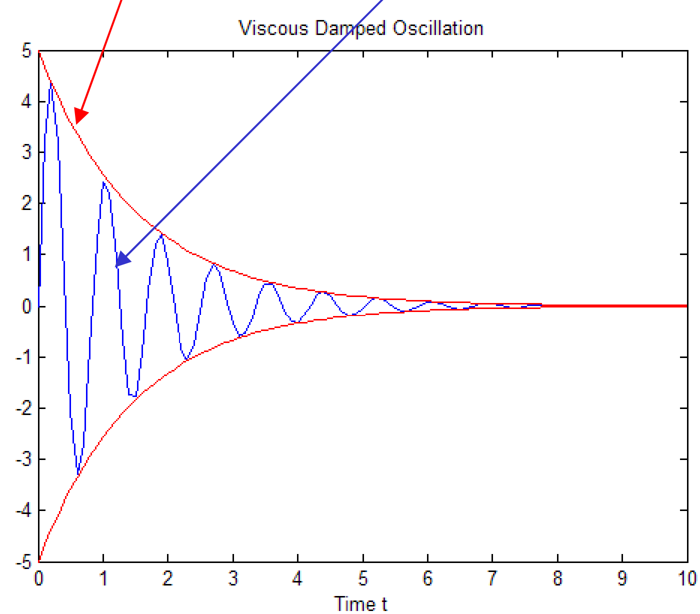
System Modeling (2^{nd} Order System)

$$m\dot{v}(t) + bv(t) = f(t)$$



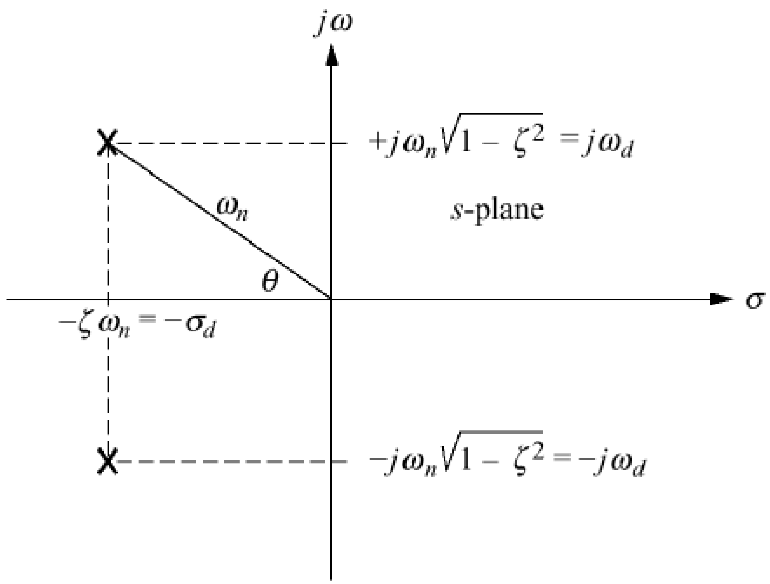
$$\underbrace{m\ddot{x}(t) + b\dot{x}(t)}\, + kx(t) = f(t)$$

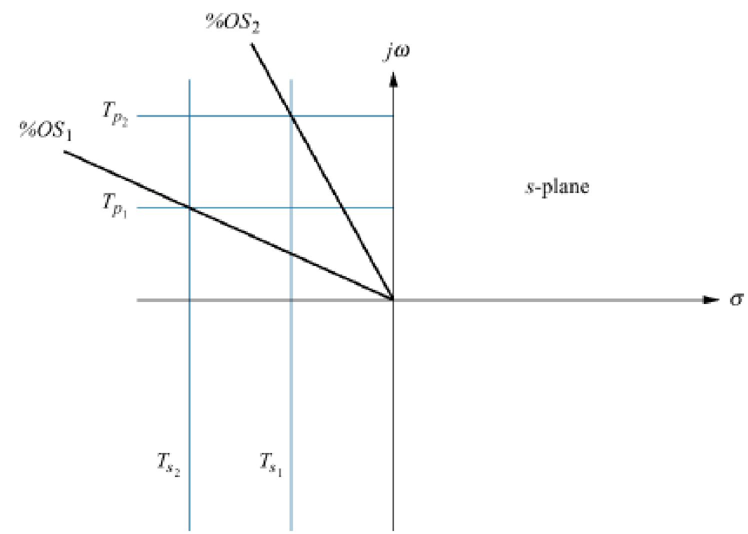
$$x = e^{-(b/2m)t}\left(Ae^{\sqrt{(b/2m)^2-(k/m)t}} + Be^{-\sqrt{(b/2m)^2-(k/m)t}} \right)$$



$$\boxed{\begin{gathered} s^2 + 2\zeta\omega_n s + \omega_n^2 \Leftrightarrow ms^2 + bs + k \\ \Rightarrow \begin{cases} \omega_n = \sqrt{\dfrac{k}{m}} \\ \zeta = \dfrac{b}{2\sqrt{km}} \end{cases} \end{gathered}}$$

2^{nd} Order System Poles





$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\boxed{s_{1,2} = -\sigma_d \pm j\omega_d = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}}$$

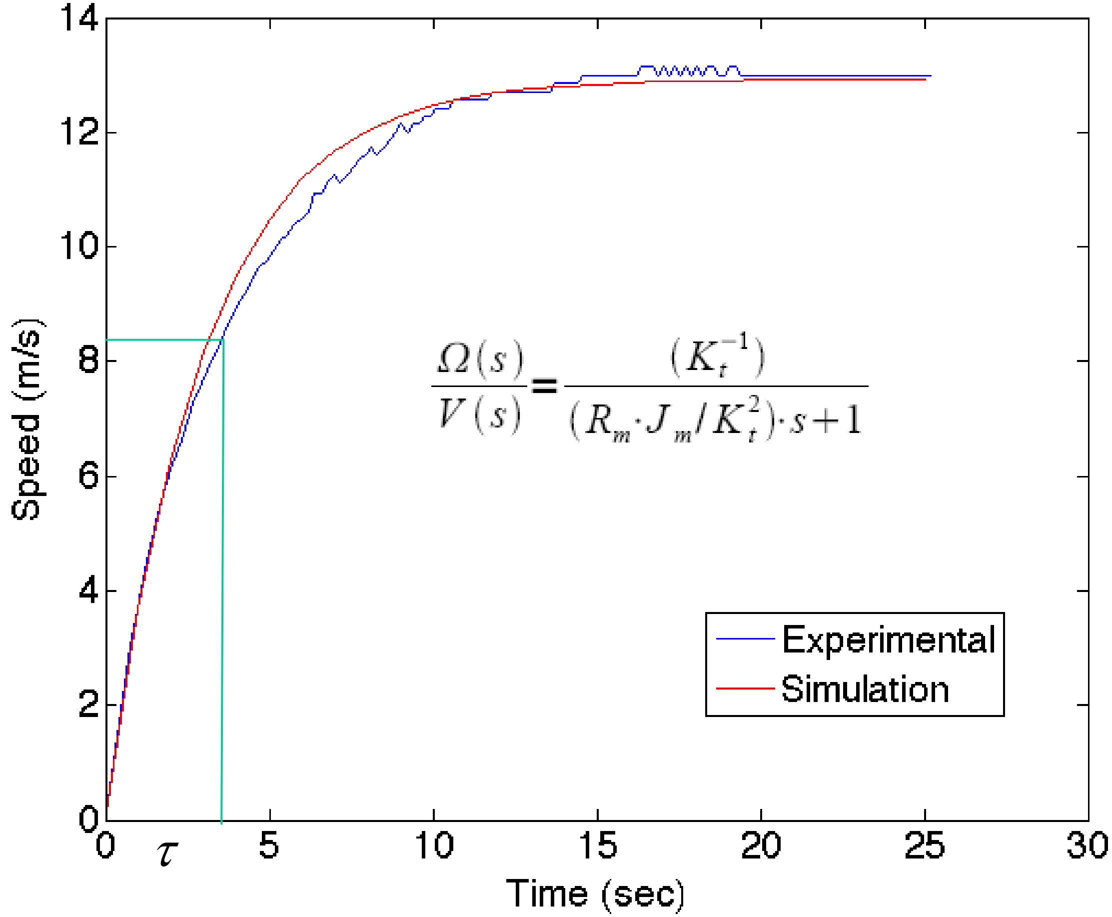
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$T_s = \frac{4}{\zeta\omega_n}$$

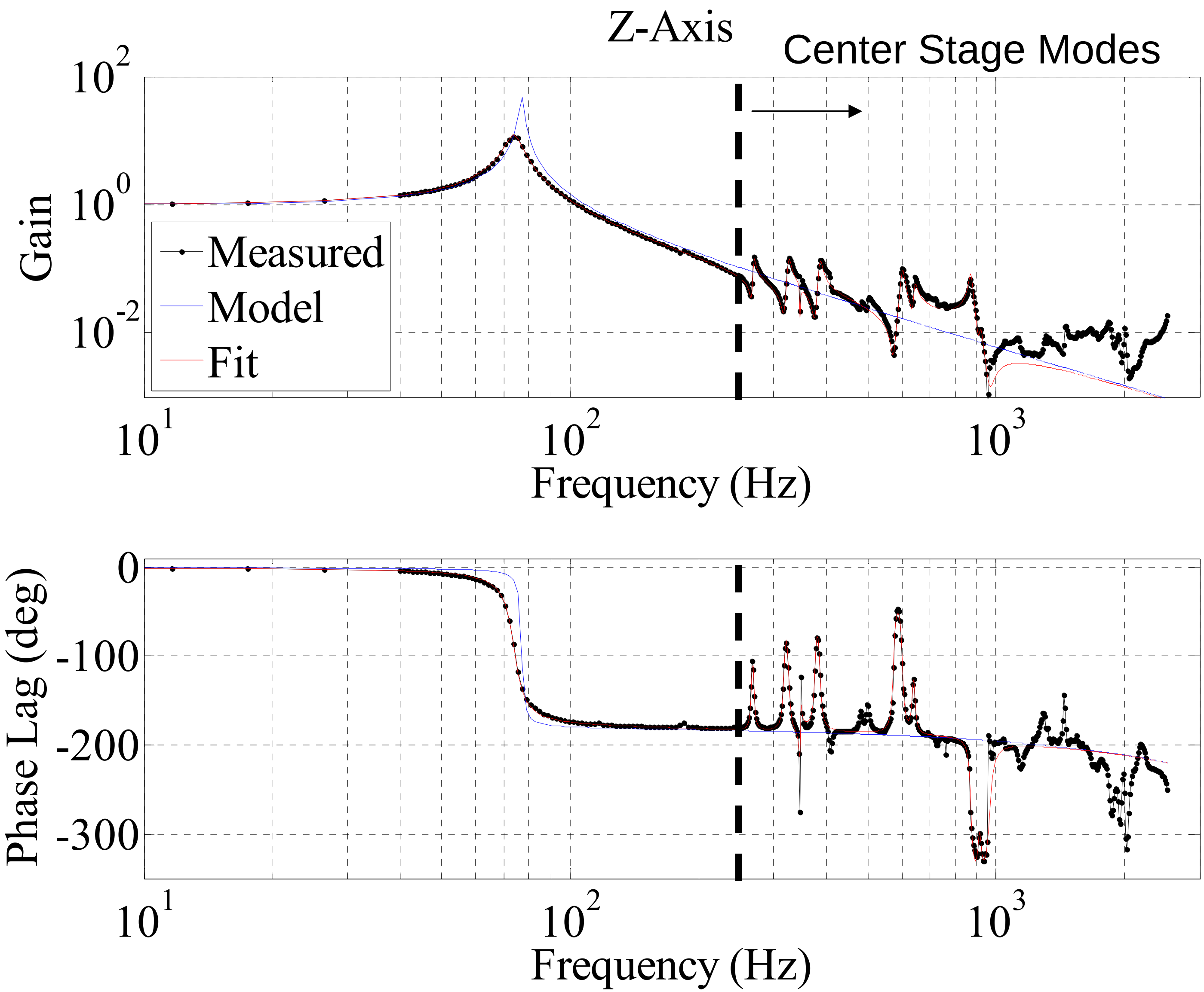
$$\%OS = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100$$

System Identification (Time Domain)

Step Input, Open Loop

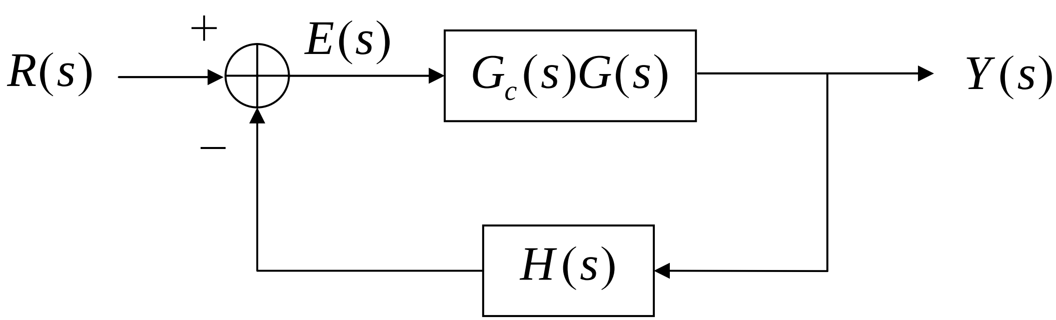


System Identification (Frequency Domain)



Closed-Loop Transfer Function

- The gain of a single-loop feedback system is given by the forward gain divided by 1 plus the loop gain.



$$\boxed{G_{cl}(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)}}$$

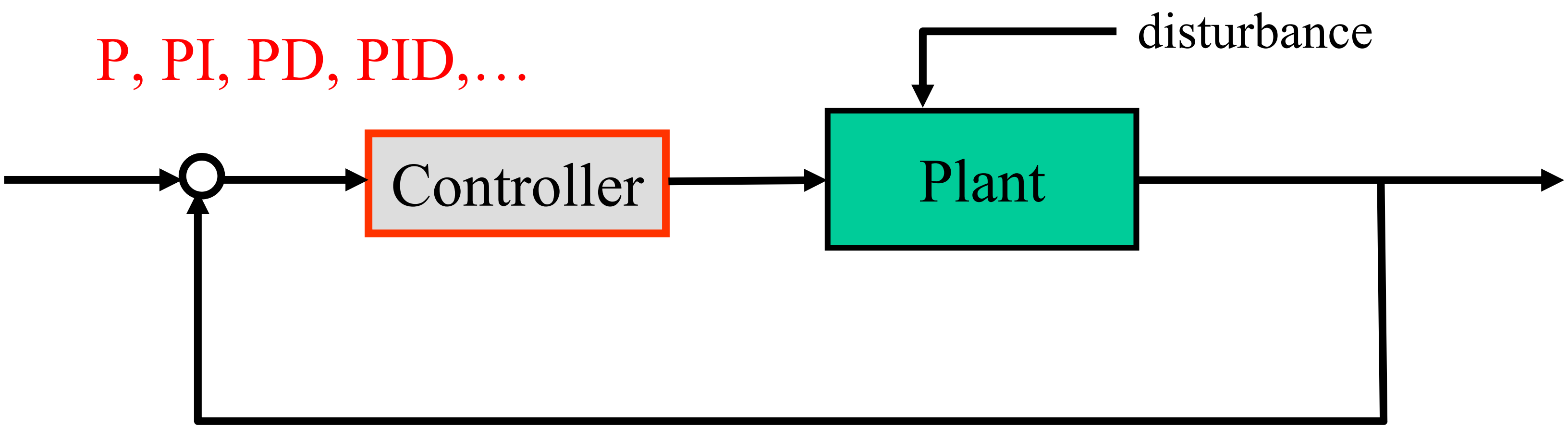
PID Controller Transfer Function

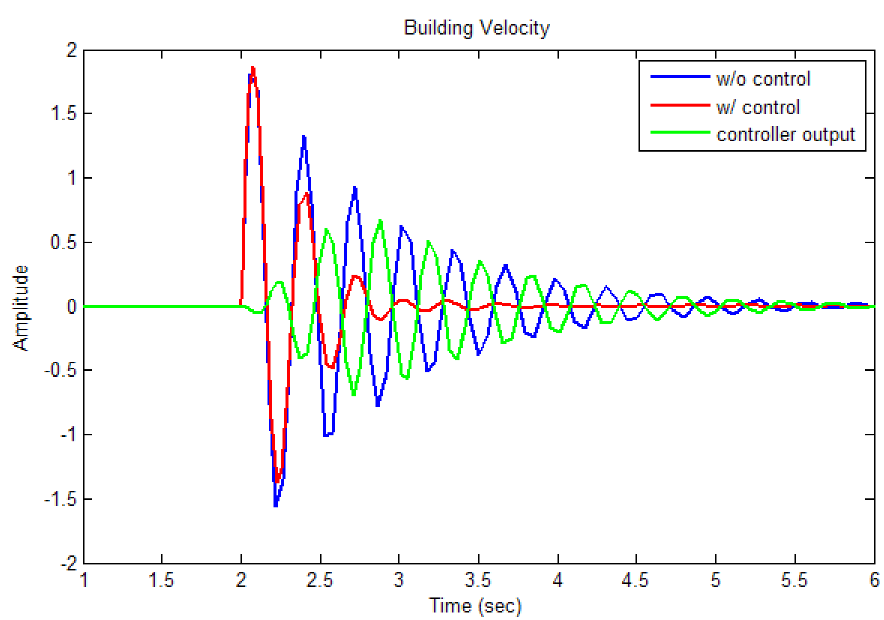
$$G_c(s) = K_p + \frac{K_i}{s} + K_d s$$

$$= \frac{K_i + K_p s + K_d s^2}{s}$$

$$= K_i \left(\frac{1 + \left(K_p \middle/ K_i \right) s + \left(K_d \middle/ K_i \right) s^2}{s} \right)$$

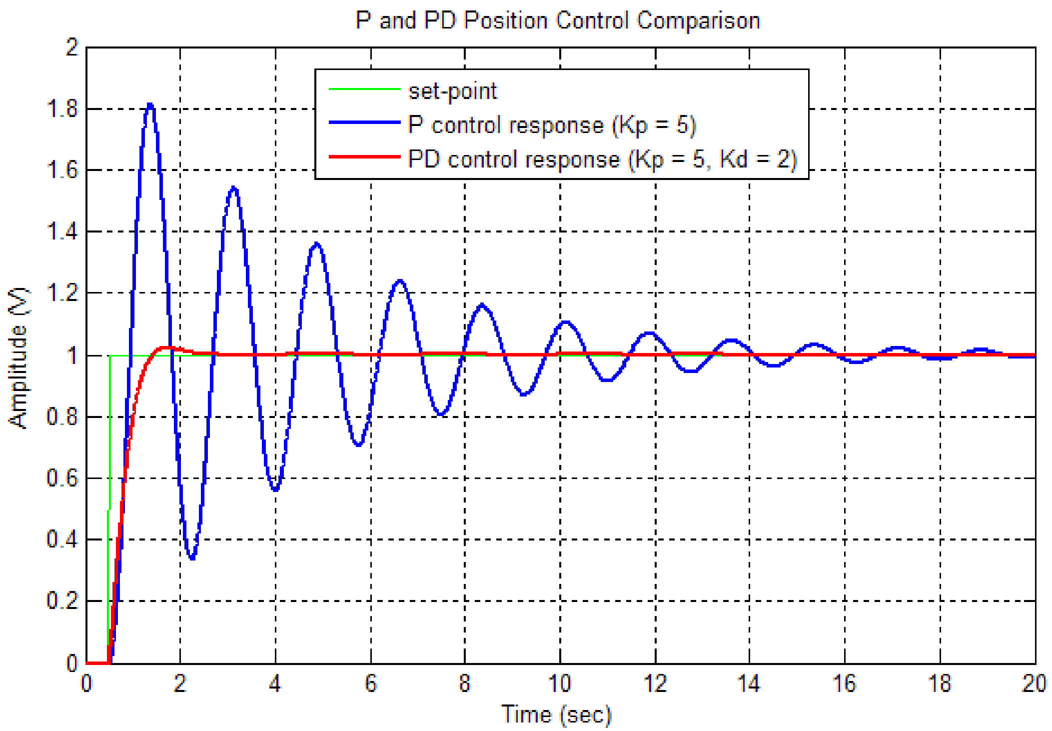
Disturbance Rejection (Active Vibration Cancellation)





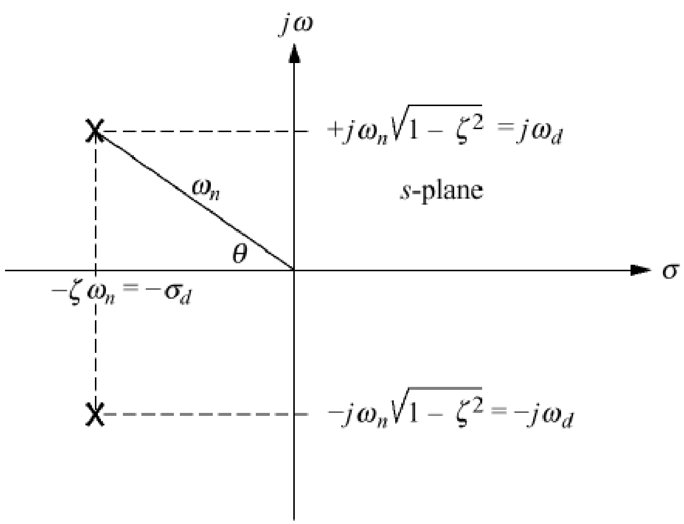
Control Actions

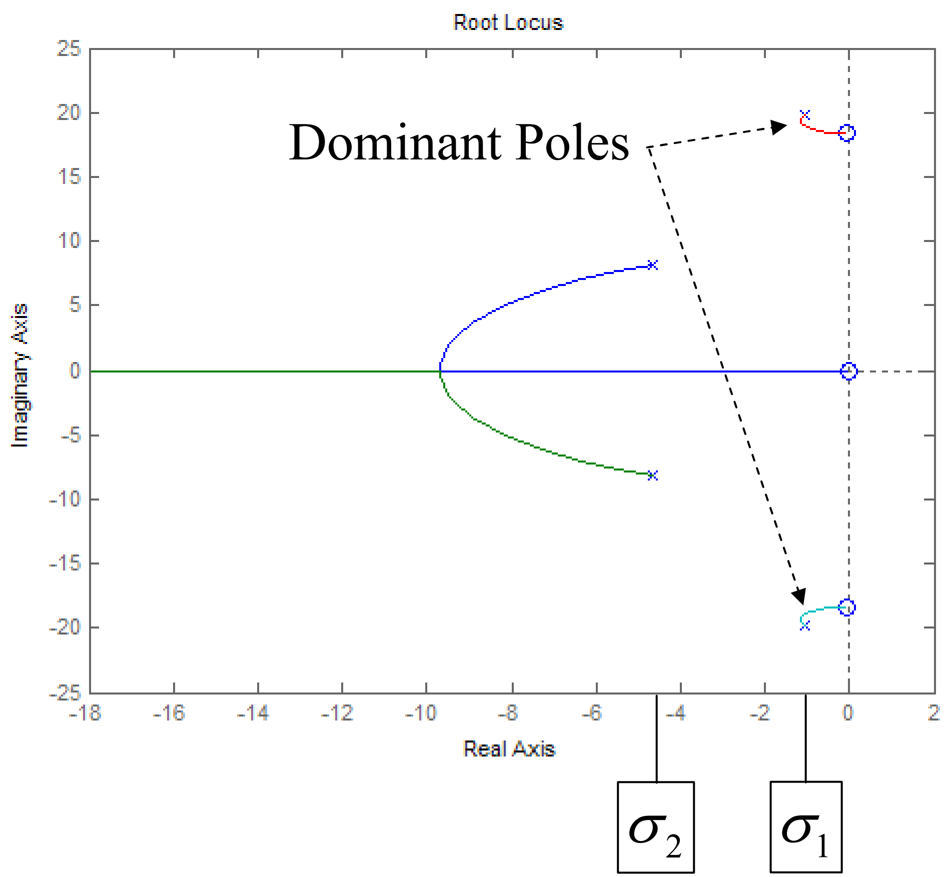
- *Proportional* – improves speed but with steady-state error
- *Integral* – improves steady state error but with less stability, overshoot, longer transient, integrator windup
- *Derivative* – improves stability but sensitive to noise



Root Locus

- Can we increase system damping with a simple proportional control ?

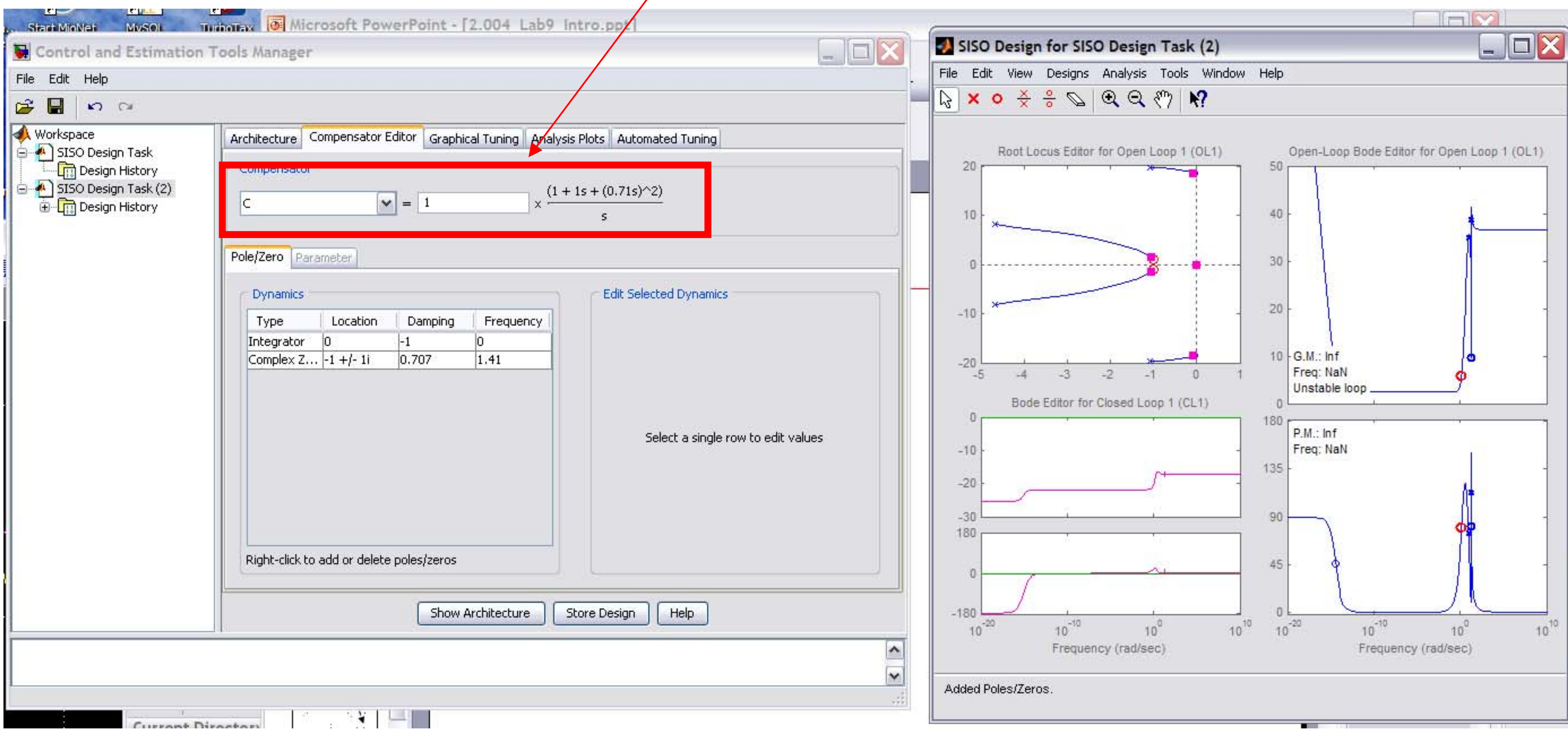




MATLAB SISO Design Tool

- MATLAB command: 'sisotool' or 'rltool'

PID Controller Transfer Function



State-Space Representation

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$$\begin{cases} \dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) \end{cases} \Rightarrow \begin{cases} sX(s) = \mathbf{A}X(s) + \mathbf{B}U(s) \\ Y(s) = \mathbf{C}X(s) + \mathbf{D}U(s) \end{cases}$$

$$(s\mathbf{I} - \mathbf{A})X(s) = \mathbf{B}U(s)$$

$$\Rightarrow X(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$

$$Y(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) + \mathbf{D}U(s)$$

$$\Rightarrow \boxed{G(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

Characteristic Polynomial

Resolvent

$$\textit{if} \quad \mathbf{D} = 0$$

$$\Rightarrow G(s) = \mathbf{C}\overbrace{(s\mathbf{I}-\mathbf{A})^{-1}}^{\text{Resolvent}}\mathbf{B} = \mathbf{C}\left[\frac{\operatorname{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}\right]\mathbf{B}$$

$$= \frac{\det(s\mathbf{I}-\mathbf{A}+\mathbf{BC}) - \det(s\mathbf{I}-\mathbf{A})}{\underbrace{\det(s\mathbf{I}-\mathbf{A})}_{\text{Characteristic polynomial}}}$$

Characteristic polynomial

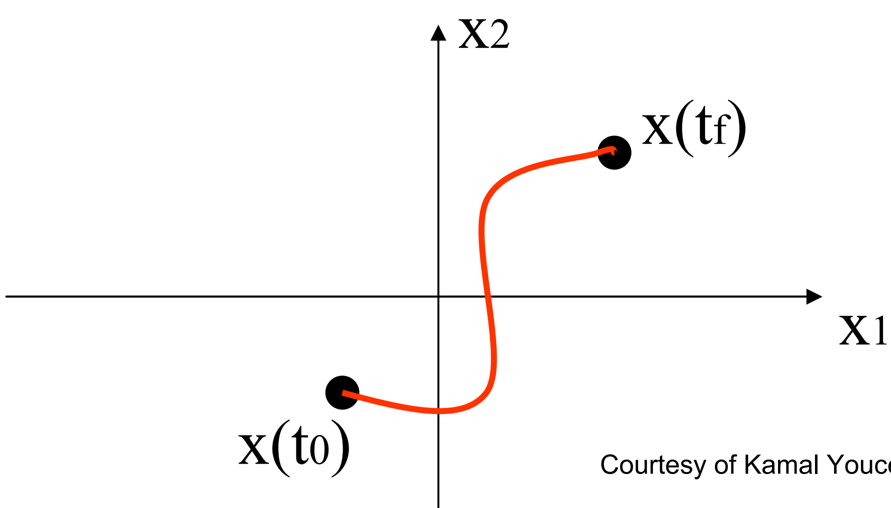
MATLAB ss2tf command uses this formula to compute the transfer function(s)

Controllability

Definition 12.1 Controllability
A system described by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))\end{aligned}$$

is said to be controllable if any initial state $\mathbf{x}(t_0)$ *can be transfered to any final state* $\mathbf{x}(t_f)$ *in a finite time* $t_f - t_0 \geq 0$ *by some piecewise continuous control signal* $\mathbf{u}(t)$*. If every state* $\mathbf{x}(t_0)$ *of the system is controllable, the system is said to be completely state controllable or simply controllable.*



Observability

Definition 12.3 Observability
A system described by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{aligned}$$

is observable if any fixed initial state $\mathbf{x}(t_0)$ *can be exactly determined from the measurements of the output* $\mathbf{y}(t)$ *and the input* $\mathbf{u}(t)$ *over a finite interval of time. If every state of the system is observable, the system is said to be completely observable or simply observable.*

Stabilizability and Detectability

Definition 12.2 Stabilizability

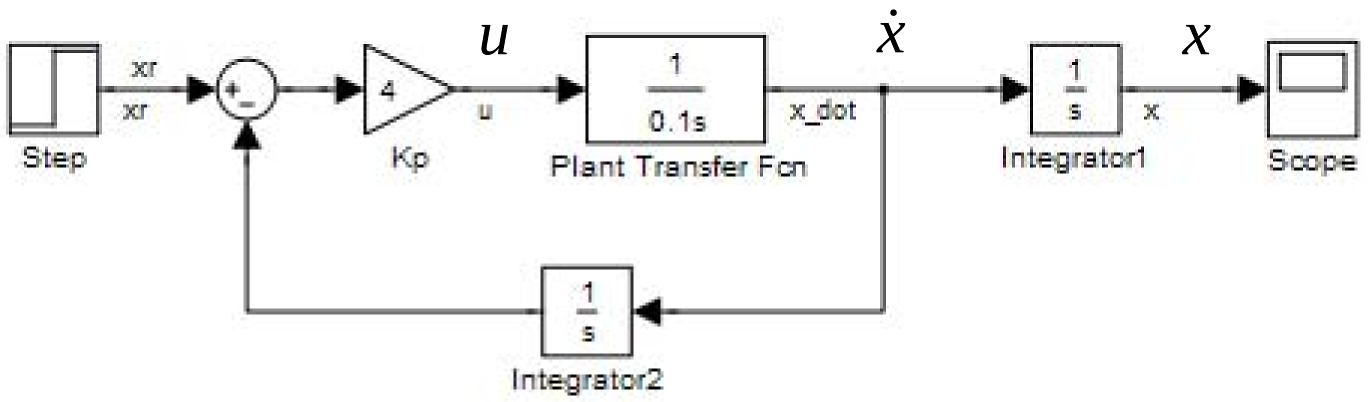
A system is said to be stabilizable if the uncontrollable modes are stable.

Definition 12.4 Detectability

A system is said to be detectable if the unobservable modes are stable.

Example

- Can we observe and/or control the position (x) of the following system?



Full-State Feedback

$$\begin{cases} \dot{x} = \mathbf{A}x + \mathbf{B}u \\ y = \mathbf{C}x \end{cases}$$

Open-loop characteristic equation: $\boxed{\det[s\mathbf{I} - \mathbf{A}] = 0}$

$$u = -\mathbf{K}x$$

$$\dot{x} = \mathbf{A}x - \mathbf{BK}x = (\mathbf{A} - \mathbf{BK})x$$

Closed-loop characteristic equation: $\boxed{\det[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})] = 0}$

Where to Place The Poles?

- Must meet the performance requirements:
  - Stability
  - Speed of response
  - Robustness
- For a given state the larger the gain, the larger the control input
- Avoid actuator saturation
- Avoid stressing the hardware (not exciting any structural modes)
- The gains are proportional to the amounts that the poles are to be moved. The less the poles are moved, the smaller the gain matrix.

Butterworth Pole Configurations

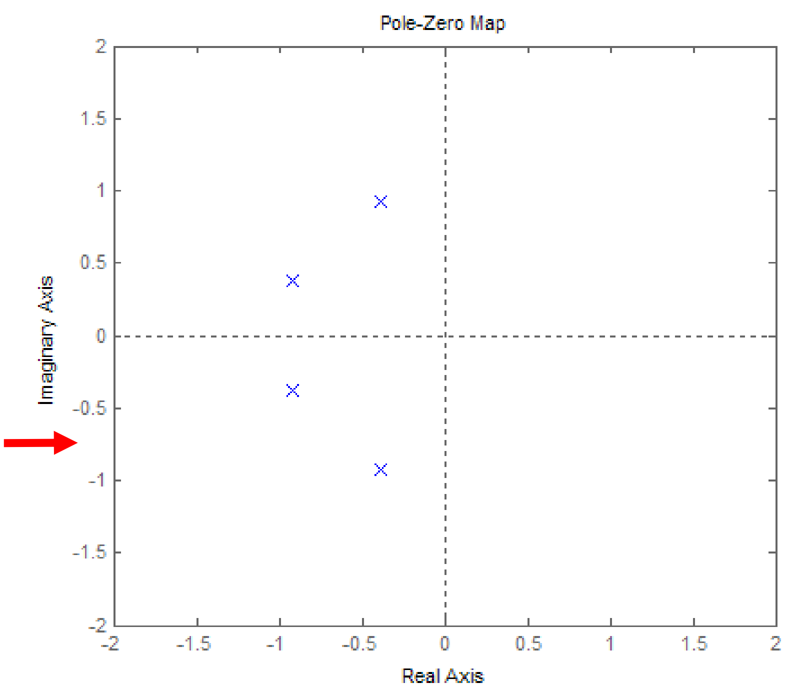
- The bandwidth of a system is governed primarily by its dominant poles (i.e., the poles w/ real parts closest to the origin)
- Efficient use of the control signal would require that all the closed-loop poles be about the same distance from the origin (a.k.a Butterworth configuration)

$$B_1(s) = s + 1$$

$$B_2(s) = s^2 + \sqrt{2}s + 1$$

$$B_3(s) = s^3 + 2s^2 + 2s + 1$$

$$B_4(s) = s^4 + 2.613s^3 + \left(2 + \sqrt{2}\right)s^2 + 2.613s + 1$$



State-Space Design Summary

- Formulate the state-space model
- Make sure the system is both controllable and observable by checking the ranks of the controllability and the observability matrices
  - Add additional actuators if necessary
  - Add additional sensors if necessary
  - Eliminate redundant states
- Select a bandwidth high enough to achieve the desired speed of response
- Keep the bandwidth low enough to avoid exciting unmodeled high-frequency modes and noise
- Place the poles at roughly uniform distance from the origin for efficient use of the control effort

Example

$$\begin{cases} \dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) = \mathbf{C}x(t) \end{cases}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 4 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Place closed-loop poles according to the Butterworth configuration

$$\det\left[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})\right] = B_2(s) = s^2 + \sqrt{2}s + 1$$

Bass-Gura formula: $\mathbf{K} = \left[\left(\mathbf{QW}\right)'\right]^{-1}\left(\hat{a} - a\right)$

Ackermann's formula: MATLAB command "acker(A,B,p)"

Example MATLAB Code

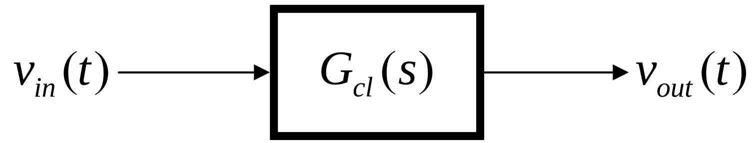
```
% 2.14/2.140 State-Space Method Example


%% Set up an SS model
A = [0 1
    4 -2];
B = [0
    1];
C = [1 1];
D = 0;


%% Convert to transfer function
[num,den] = ss2tf(A,B,C,D,1);
sys_tf = tf(num,den)
zpk(sys_tf)
pzmap(sys_tf)
hold
```

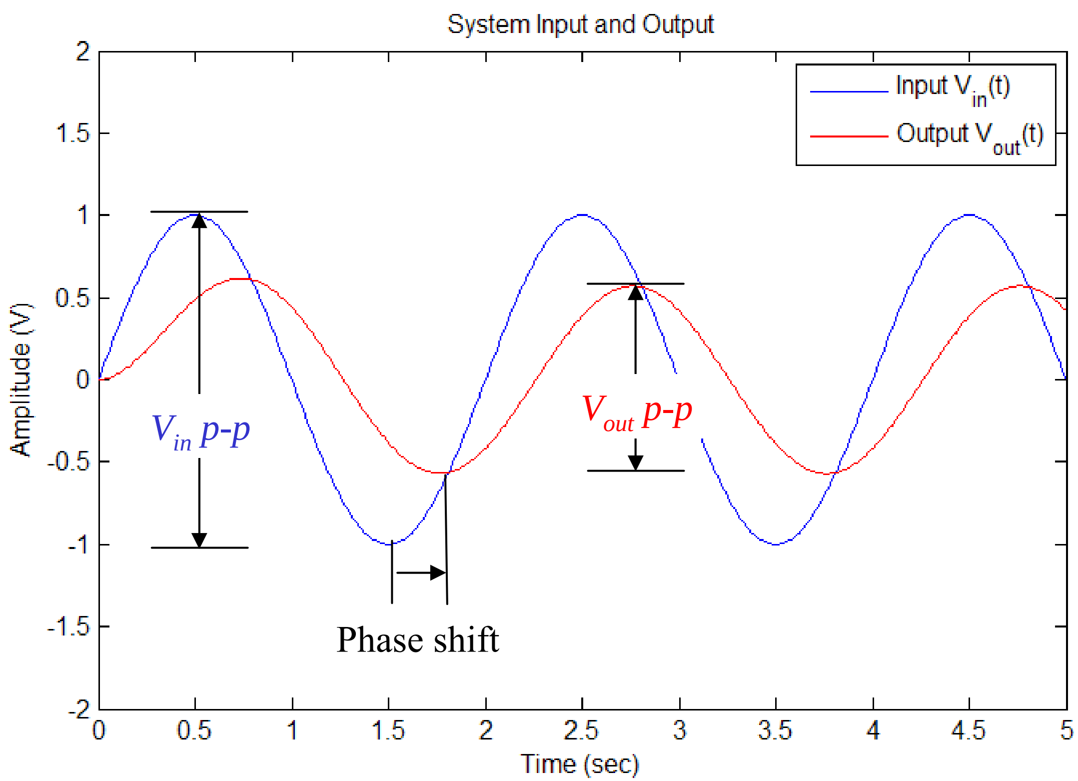
```
%% Test controllability and observability
CtrlTestMatrix = ctrb(A,B)
rank(CtrlTestMatrix)

ObsrbTestMatrix = obsv(A,C)
rank(ObsrbTestMatrix)

%% Place the poles to Butterworth configuration
p = roots([1 sqrt(2) 1])
% K = acker(A,B,p) % this method is not numerically
reliable and starts to break down rapidly for problems of
order greater than 5
K = place(A,B,p)

% check the closed-loop pole locations
eig(A-B*K)
pzmap(1,poly(eig(A-B*K)))
```

Frequency Design Methods

- Loop shaping
- Bode, Nyquist
- Crossover frequency
- Closed-loop bandwidth
- Phase margin

Frequency Response (Gain and Phase)



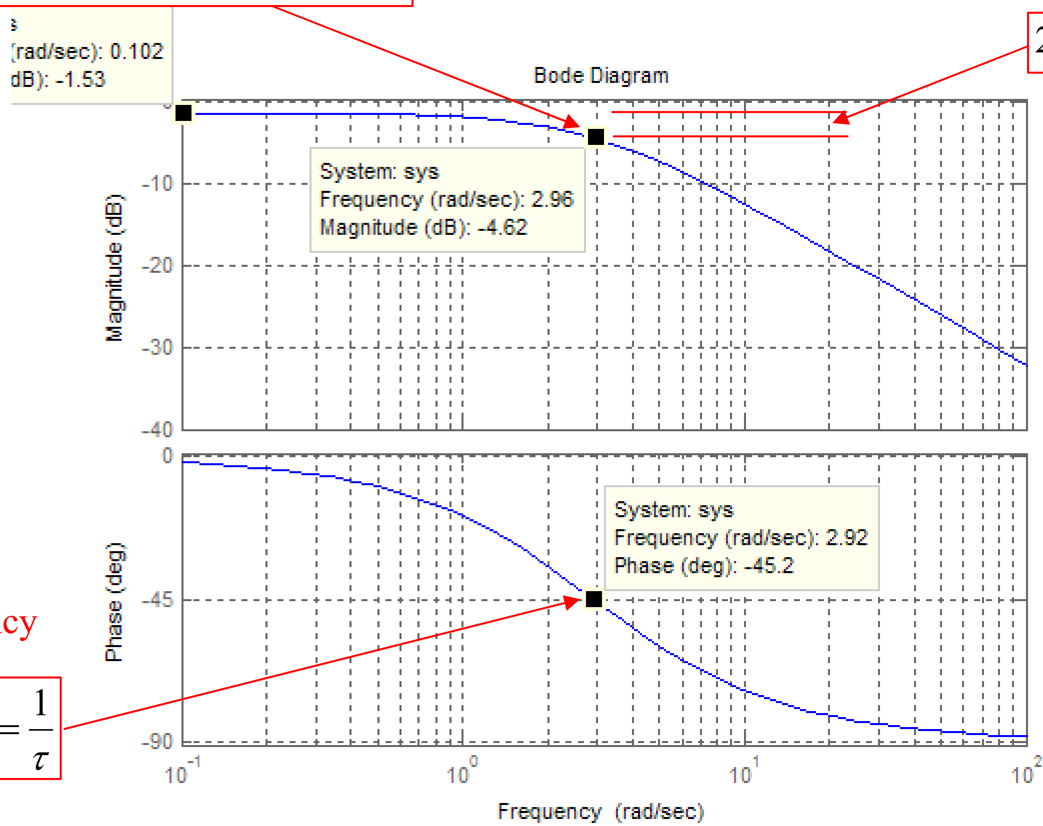


Frequency Response (Bode Plot)

- The frequency response of a system is typically expressed as a Bode plot.

$$M_c = M_0 \cdot \sin\left(\frac{\pi}{4}\right) = 0.707 \cdot M_0$$

$$20 \cdot \log_{10}(0.707) = 3dB$$



Gain Plot

Phase Plot

Cutoff frequency

$$\omega_c = 2\pi f_c = \frac{1}{\tau}$$

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