

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering
2.004 Dynamics and Control II
Fall 2007

Problem Set #6

Solution

Posted: Friday, Oct. 26, '07

1. Starting with the Supplement to Lecture 12 (“Supplement” for short,) modify the time–domain torque balance equation (3) on the motor shaft and force balance equation (5) on the mass to include the compliances.

Answer: Since we have compliance terms in the system, the spring torque T_c and force F_c are

$$\begin{aligned}T_C(t) &= C_r\theta(t), \\F_C(t) &= C_t x(t).\end{aligned}$$

Note that the spring torque and force are proportional to displacement $\theta(t)$ and $x(t)$ respectively, which are time integrals of $\omega(t)$ and $v(t)$. Thus equations (3) and (5) will be

$$\begin{aligned}K_m i(t) &= J\dot{\omega}(t) + D\omega(t) + C_r \int_0^t \omega(\tau) d\tau + T_p(t), \\F_p(t) &= M\dot{v}(t) + f_v v(t) + C_t \int_0^t v(\tau) d\tau.\end{aligned}$$

2. Laplace transform your new torque/force–balance equations and repeat the steps leading to equation (13) of the Supplement. The KVL equation should be identical to (1–2) since we are neglecting the inductance, as we did in the Supplement. Obtain the transfer function $V(s)/V_s(s)$ of the compliant system without feedback. After substituting $C_r = 2\text{N} \cdot \text{m}/\text{rad}$, $C_t = 10\text{N}/\text{m}$ and the remaining numerical values from the Supplement, your transfer function should be

$$\frac{0.3162s}{s^2 + 2s + 3}$$

You should use this transfer function in the rest of the problem set, even if you can’t get your algebra from question 1 to match it.

Answer: Applying the Laplace transform to the previous results, we obtain

$$\begin{aligned}K_m I(s) &= Js\Omega(s) + D\Omega(s) + C_r \frac{\Omega(s)}{s} + T_p(s), \\F_p(s) &= MsV(s) + f_v V(s) + C_t \frac{V(s)}{s}.\end{aligned}$$

Using the above two relations, we obtain the Laplace transform of the current $i(t)$ as

$$I(s) = \left(\frac{J + r^2 M}{rK_m} s + \frac{D + r^2 f_v}{rK_m} + \frac{C_r + r^2 C_t}{rK_m} \frac{1}{s} \right) \Omega(s).$$

Combining this result with KVL,

$$V_s(s) = \left(\frac{J + r^2 M}{rK_m} s + \frac{D + r^2 f_v}{rK_m} + \frac{C_r + r^2 C_t}{rK_m} \frac{1}{s} \right) RV(s) + K_v \frac{V(s)}{r},$$

we end up with the transfer function given by

$$\frac{V(s)}{V_s(s)} = \frac{\frac{rK_m}{R(J + r^2 M)} s}{s^2 + \frac{D + r^2 f_v + K_m K_v / R}{J + r^2 M} s + \frac{C_r + r^2 C_t}{J + r^2 M}}.$$

Substituting the numerical values, we find

$$G(s) = \frac{0.3162s}{s^2 + 2s + 3}.$$

- 3.** Now consider the complete feedback system, which includes the differential amplifier with reference voltage input and velocity feedback. Write expressions for the open loop transfer function and the closed-loop transfer function of this system with the feedback gain K as a variable.

Answer:

Open loop transfer function $G(s)$ is

$$\frac{0.3162s}{s^2 + 2s + 3}.$$

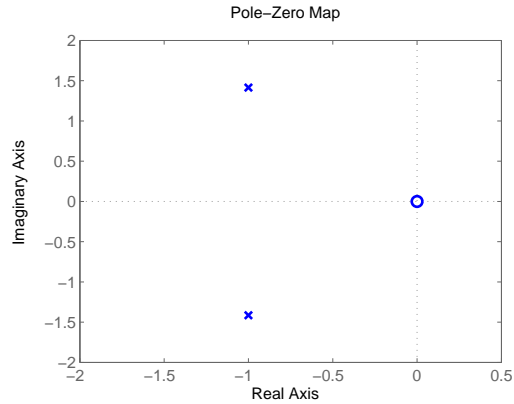
Closed loop transfer function is

$$\frac{0.3162Ks}{s^2 + (2 + 0.3162K)s + 3}.$$

- 4.** What is the order, type, and nature (*i.e.*, overdamped, undamped, etc.) of this system? Back up your answer with a sketch of the open-loop zeros and poles.

Answer: Obviously this system has two poles, so it is 2nd order system. The number of pure integrators is zero, thus it is Type 0. (The steady-state error is non-zero for step input and zero for ramp input.) The open loop system is underdamped, but the closed loop system transitions from underdamped to overdamped as the gain K increases because the poles are

$$p_{1,2} = \left\{ -(2 + 0.3162K) \pm \sqrt{(2 + 0.3162K)^2 - 12} \right\} / 2.$$



5. What is the steady-state velocity of this system? What is the steady-state error? Provide an explanation based on the *physics* of the system.

Answer: From the final value theorem, the steady-state velocity is

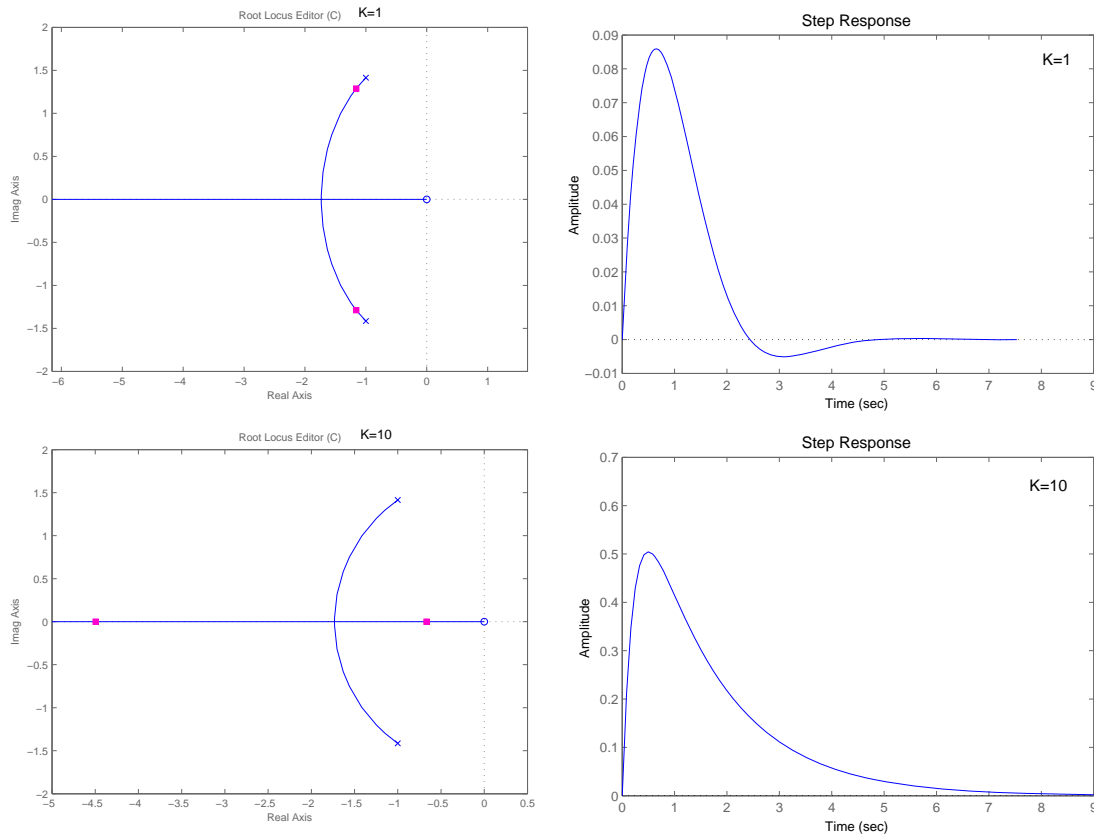
$$v_{\infty} = \lim_{s \rightarrow 0} s \left(\frac{1}{s} \right) \frac{0.3162Ks}{s^2 + (2 + 0.3162K)s + 3} = 0.$$

Therefore the steady state error is

$$e_{\infty} = \lim_{s \rightarrow 0} s \left(\frac{1}{s} \right) \frac{1}{1 + KG(s)} = \lim_{s \rightarrow 0} \frac{s^2 + 2s + 3}{s^2 + (2 + 0.3162K)s + 3} = 1.$$

The spring force/torque tries to get the system back to the initial (neutral) position. Once the step input drives the motor, then $\theta(t)$ and $x(t)$ increase, which leads to increase on spring force/torque. Since the magnitude of the step input is constant, the driving torque and the spring force/torque will be competing with each other and finally balance at steady-state. The mass will then stop moving because the driving torque and spring force/torque balance each other. So the steady-state velocity is zero, but the system has some non-zero displacement at steady-state.

6. Specify your system in the SISO tool and use the **Analysis**→**Response to step command** menu choice to plot the pole locations and step response for $K = 1$ and $K = 10$. Confirm that the step responses match qualitatively the pole locations and your steady-state prediction from question 5.



As pointed out in question 4, the system transitions from under- to overdamped as the gain K increases. However, the steady-state velocity is zero, independent of K .

When $K = 1$, the two poles have imaginary parts and the system is underdamped. Thus we expect overshoot before the steady-state is reached. When $K = 10$, the system has two real negative poles and is overdamped. The form of the step response is $ae^{-p_1t} + be^{-p_2t}$, which does not yield overshoot.

Small gain $K = 1$ yields oscillatory motion to reach the steady-state because the spring force/torque dominates the amplified error signal. However, interestingly, large gain $K = 10$ does not generate overshoot because the amplified error signal then dominates the spring force/torque, so the system behaves as overdamped. In other words, large gain effectively increases the system damping. This can be understood by looking at the denominator of the transfer function. Only the dissipation (first-order) s term is dependent on gain K , natural frequency is always $\sqrt{3}$ (rad/s) and damping ratio $\zeta = (2 + 0.3162K)/2/\sqrt{3}$. Therefore the greater the gain K , the greater the damping ratio ζ .

7. Explain the shape of the root locus that you see in the SISO tool panel based on the root locus sketching rules that we covered in class (but don't attempt any quantitative verification of the root locus features.)

Answer: It has two poles at $s = -1 \pm j\sqrt{2}$ and two zeros at $s = 0$ and $s = \infty$.

By rules 1 and 4, it has two branches departing from the two poles and ending at the two zeros. By rule 3 about real axis segment, one segment should be joining negative infinity to the zero at $s = 0$. Since two branches start from the two poles, they have to be joined to the segment on the real axis, so that one branch goes to zero at $s = 0$ and the other one goes to the negative infinity.

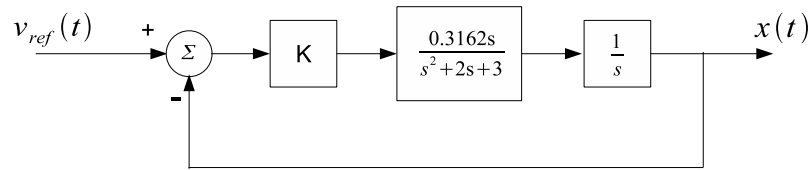
We can (but you didn't have to in your solution) also compute the breakaway point by rule 6.

$$\frac{dK(\sigma)}{d\sigma} = \frac{d}{d\sigma} \left(\frac{\sigma^2 + 2\sigma + 3}{\sigma} \right) = \frac{(2\sigma + 2)\sigma - (\sigma^2 + 2\sigma + 3)}{\sigma^2} = \frac{\sigma^2 - 3}{\sigma^2} = 0$$

Thus the breakaway point $\sigma = -\sqrt{3}$.

8. Now add an integrator to the forward path of the closed-loop system. Sketch the new block diagram and interpret the operation of the new system with the integrator. What would change in the physical implementation of the system? What is the meaning of the feedback signal and the reference voltage?

Answer:



Due to the integrator, the old output $v(t)$ is integrated, which means that the new output is the position $x(t)$ of the mass. The position $x(t)$ is fed-back to the differential input of the op-amp. We don't need to change any system implementation except replace the tachometer with a position meter. Since we choose position as the system output, V_{ref} now indicates the reference (or desired) position.

9. What is the steady-state output of the new system, as function of feedback gain K ? Is the result consistent with the system type? Is the result consistent with your answers to questions 5 and 8?

Answer: Including the integrator in the open-loop transfer function, we obtain the closed-loop transfer function as

$$\frac{X(s)}{V_{ref}(s)} = \frac{0.3162K}{s^2 + 2s + (3 + 0.3162K)}$$

The steady-state output of the system, position of the mass, is given by the final value theorem.

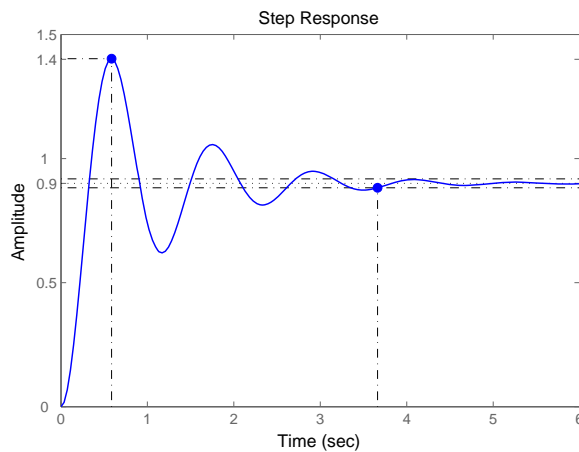
$$x_\infty = \lim_{s \rightarrow 0} s \left(\frac{1}{s} \right) \frac{0.3162K}{s^2 + 2s + (3 + 0.3162K)} = \frac{0.3162K}{3 + 0.3162K}$$

The integrator that we added cancels the zero of the system (s in the numerator). Hence the number of pure integrators is still zero, and the system is still Type 0. It yields non-zero steady-state error for step input.

Once the step input applied, the motor starts to move the mass to the desired position, which is decided by the motor torque and spring force. At steady-state, the motor stops rotating. The steady-state position and error are dependent on the gain K . Also note that the poles are $s = -1 \pm j\sqrt{2 + 0.3162K}$, so the system is always underdamped regardless of K (of course we assume positive gain only).

10. Define the new system in the SISO tool (do *not* use the SISO tool's option of adding an integrator to the compensator block; instead, define the open-loop transfer function anew, including the integrator, and import it as the plant.) Use the **Analysis**→**Response to step command** menu choice to plot the pole locations and step response for $K = 85.4$. Find the steady-state output from the numerical result and compare it with your analytical expression from question 9. Find also the percent overshoot.

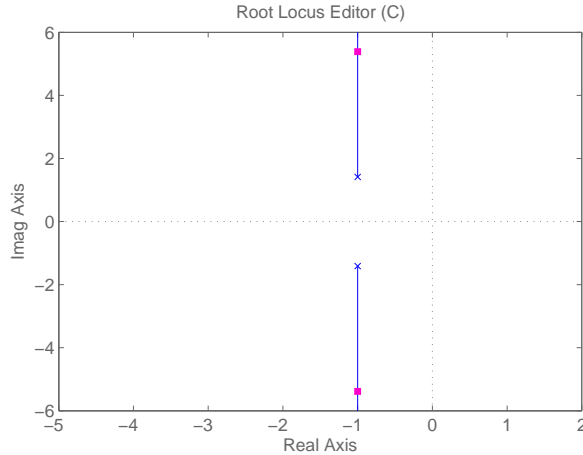
Answer:



From the previous result in question 9, the steady-state output is 0.9 when $K = 85.4$, which is consistent with the plotted result. The percent overshoot is 55.8%.

11. Explain the shape of the new root locus from question 10 based on the root locus sketching rules.

Answer:



With the integrator, we have only two poles. The root locus will start from $s = -1 \pm j\sqrt{2}$ (open-loop poles). This is consistent with what we expected from question 9, where we found that the system is always underdamped. There are two zeros at infinity as we can see by writing the open-loop transfer function as

$$\frac{0.3162}{s^2 + 2s + 3} \approx \frac{0.3162}{s \cdot s} \Big|_{s \rightarrow \infty}.$$

Therefore, the closed-loop poles must go to ∞ as $K \rightarrow \infty$ along the two asymptotes at angles (Rule 5)

$$\theta_a = \frac{(2m+1)\pi}{2-0} = \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}.$$

The asymptote real axis intercept is (also by Rule 5)

$$\frac{(-1 + j\sqrt{2 + 0.3162K}) + (-1 - j\sqrt{2 + 0.3162K})}{2 - 0} = -1.$$

The asymptote intercept coincides with the real parts of the complex-conjugate poles; therefore the RL overlaps the asymptotes.

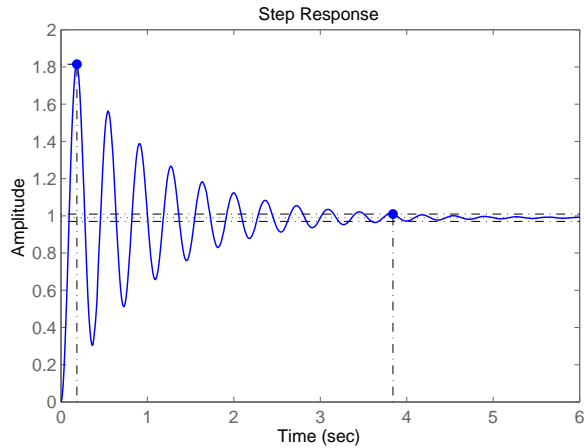
- 12.** Use the expression you derived in question 9 to find the feedback gain K that would be required for the steady-state output to equal 0.99. Then modify the controller gain in the SISO tool to equal the K value that you obtained and compute the step response. Verify that the steady-state output that MATLAB produced matches the value of 0.99 and find the percent overshoot.

Answer:

From the result of question (9), the steady-state output is given by

$$x_\infty = \frac{0.3162K}{3 + 0.3162K} = 0.99.$$

Therefore $K = 939.3$ and its step response is plotted as



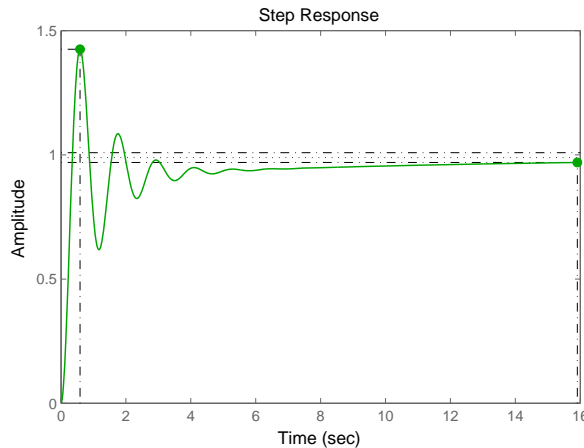
The percent overshoot is 83.3%.

13. Define a new transfer function

$$85.4 \times \frac{s + 0.1}{s + 0.01}$$

and import it as the Compensator C Block in the SISO tool. Obtain the new step response and compare it with the step responses from questions 10 and 12. What do you observe? (We will learn later in class that the new compensator represents a type of *integral control* called a *lag compensator*. The lag compensator would require replacement of the differential amplifier by a more complicated circuit.)

Answer:



The steady-state output is still 0.99 and the percent overshoot is 43.3%. Please note the time scale. After the initial transient response, the step response is slightly increasing and eventually gets to steady-state. Note that we achieve 0.99 steady-state though we use the same gain as question 10 (where the steady-state value was 0.9). This choice of compensator also results in lower percent overshoot. Without the lag compensator, you need a huge gain ($K = 939.3$) to obtain the same steady-state (0.99), as we saw in question 12.

For visual comparison, we plotted the two responses below.

