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2.004 Dynamics and Control II
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Lecture 34[1]

Reading:

- Nise: 10.1

- Class Handout: Sinusoidal Frequency Response

1 Bode Plots (continued

In Lecture 33 we developed the following asymptotic Bode Plot s for low-order systems:

Pole/Zero at the Origin:

$$H(s) = \frac{1}{s} \quad \text{and} \quad H(s) = s$$
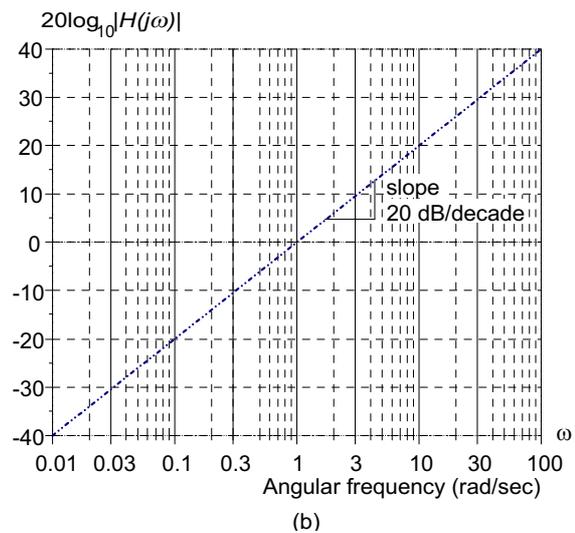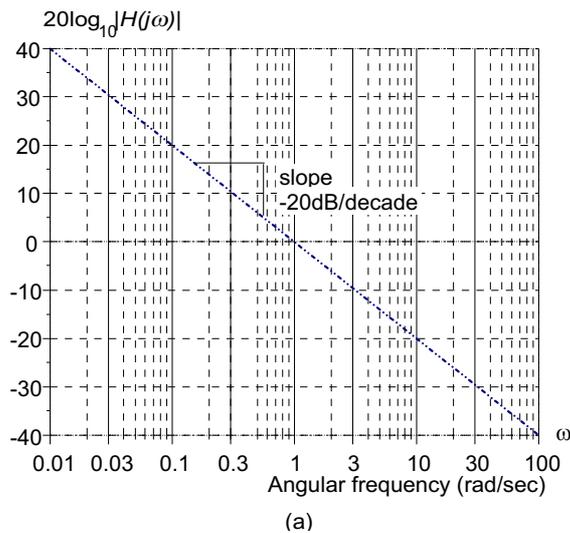
Single Real Pole/Zero:

$$H(s) = \frac{1}{\tau s + 1} \quad \text{and} \quad H(s) = \tau s + 1$$
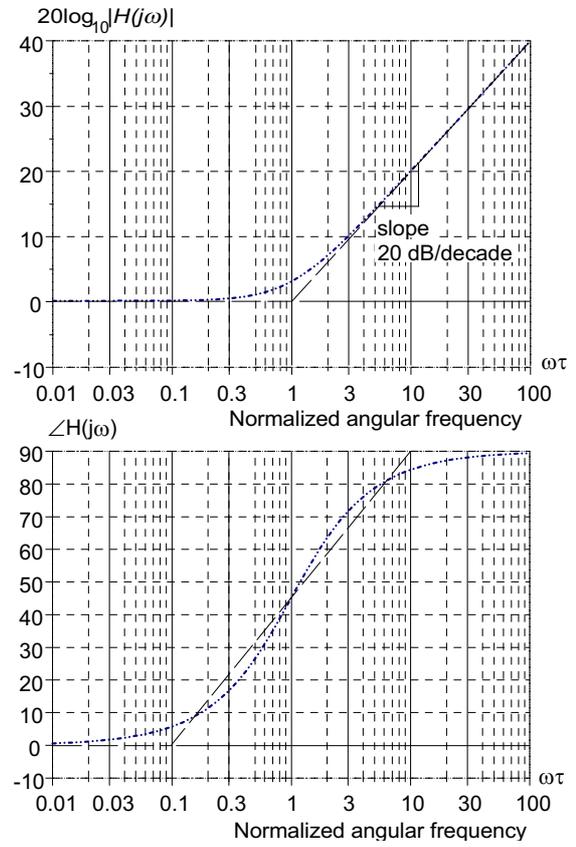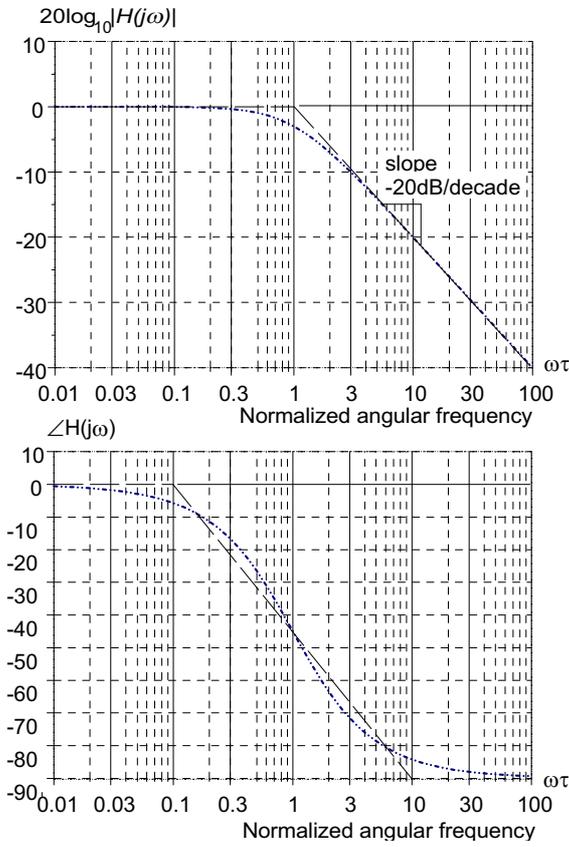
Complex Conjugate Real Pole/Zero pair:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \text{and} \quad H(s) = \frac{1}{\omega_n^2}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)$$
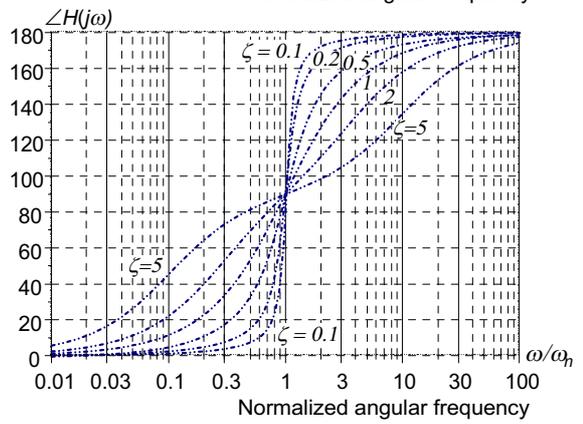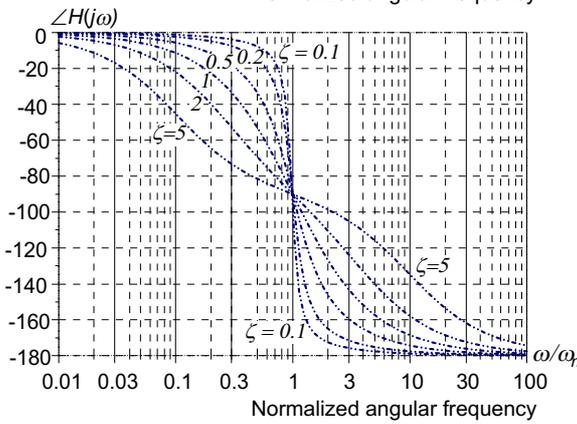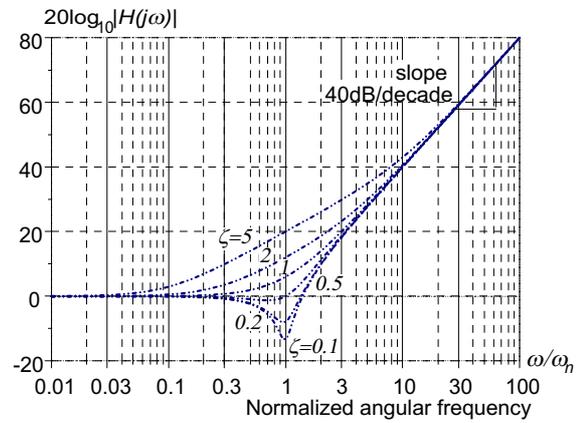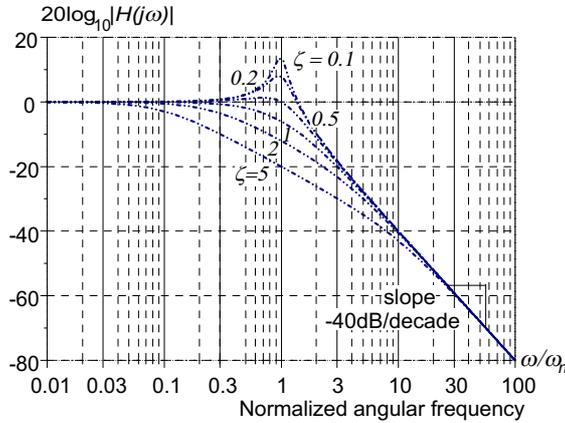
(a)

(b)

Description	Transfer Function	Break Frequency (radians/sec.)	High Frequency Slope (dB/decade)
Constant gain	K	-	0
Pole at the origin	$\dfrac{1}{s}$	-	-20
Zero at the origin	s	-	+20
Real pole	$\dfrac{1}{\tau s + 1}$	$1/\tau$	-20
Real zero	$(\tau s + 1)$	$1/\tau$	+20
Conjugate poles	$\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$	ω_n	-40
Conjugate zeros	$\dfrac{1}{\omega_n^2}(s^2 + 2\zeta\omega_n s + \omega_n^2)$	ω_n	+40

1.1 Bode Plots of Higher Order Systems

If a system with transfer function $H(s) = KN(s)/D(s)$ is expressed as a product of the terms in the table above, that is

$$
\begin{aligned}
H(s) &= K\frac{N(s)}{D(s)} \\
&= K'(N_1(s)\ldots N_m(s)) \times \left(\frac{1}{D_1(s)}\cdots\frac{1}{D_n(s)}\right)
\end{aligned}
$$

where the factors $N_i(s)$ are first- or second-order zero terms, and the $D_i(s)$ are pole terms, and K' is a modified constant factor. For example

$$
\begin{aligned}
H(s) &= \frac{10(s+3)}{(s+0.5)(s+5)} = K'N_1(s)\frac{1}{D_1(s)}\frac{1}{D_2(s)} \\
&= 12 \times \left(\frac{1}{3}s+1\right) \times \frac{1}{2s+1} \times \frac{1}{0.2s+1}.
\end{aligned}
$$

When complex numbers are represented in polar form, the magnitude of a product is the product of the component magnitudes, and the angle of a product is the sum of the component angles, the frequency response may be expressed in terms of its magnitude and phase functions:

$$
\begin{aligned}
|H(j\omega)| &= K' \times |N_1(j\omega)| \times \ldots \times |N_m(j\omega)| \times \left|\frac{1}{D_1(j\omega)}\right| \times \ldots \times \left|\frac{1}{D_n(j\omega)}\right| \\
\angle H(j\omega) &= \angle N_1(j\omega) + \ldots + \angle N_m(j\omega) + \angle\frac{1}{D_1(j\omega)} + \ldots + \angle\frac{1}{D_n(j\omega)}
\end{aligned}
$$

The logarithm of a product is the sum of the logarithms of its factors, so that

$$
\boxed{
\begin{aligned}
\log|H(j\omega)| &= \log K' + \log|N_1(j\omega)| + \ldots + \log|N_m(j\omega)| + \log\left|\frac{1}{D_1(j\omega)}\right| + \ldots + \log\left|\frac{1}{D_n(j\omega)}\right| \\
\angle H(j\omega) &= \angle N_1(j\omega) + \ldots + \angle N_m(j\omega) + \angle\frac{1}{D_1(j\omega)} + \ldots + \angle\frac{1}{D_n(j\omega)}
\end{aligned}
}
$$

which express the overall magnitude and phase responses as a *sum of component responses* of first and second-order elementary "building blocks". In practice Bode plots are constructed by graphically adding the individual response components. Given the transfer function $H(s)$ of a linear system, the following steps are used to construct the Bode magnitude plot:

1. Factor the numerator and denominator of the transfer function into the constant, first-order and quadratic terms in the form described in the previous section.

2. Identify the break frequency associated with each factor.

3. Plot the asymptotic form of each of the factors on log–log axes.

4. Graphically add the component asymptotic plots to form the overall plot in straight line form.

5. "Round out" the corners in the straight line approximate curve by hand, using the known values of the responses at the break frequencies ($\pm$3dB for first-order sections, and dependent upon ζ for quadratic factors).

The phase plot is constructed by graphically by adding the component phase responses. The individual plots are drawn, either as the piece-wise linear approximation for the first-order poles, or in a smooth form from the exact plot, and then these are added to find the total phase shift at any frequency.

■ Example 1

Plot the Bode magnitude and phase plots of a first-order system described by the transfer function

$$H(s) = \frac{10(s+1)}{s+10}$$

Solution: The transfer function is rewritten in terms of unity-gain blocks

$$H(s) = 10 \times (s+1) \times \frac{1}{10} \times \frac{1}{0.1s+1} = (s+1) \times \frac{1}{0.1s+1}$$

with two component terms:

1. A single real zero term $H_1(s) = (s+1)$, with a break frequency of $\omega = 1$ radians/sec.

2. A single real pole term $H_2(s) = \frac{1}{0.1s+1}$, with a break frequency of $\omega = 10$ radians/sec.

The component terms are plotted and are added together to determine the total response for a frequency range of 0.01 to 1000 radians/sec. in the magnitude and phase plots below.

34–5
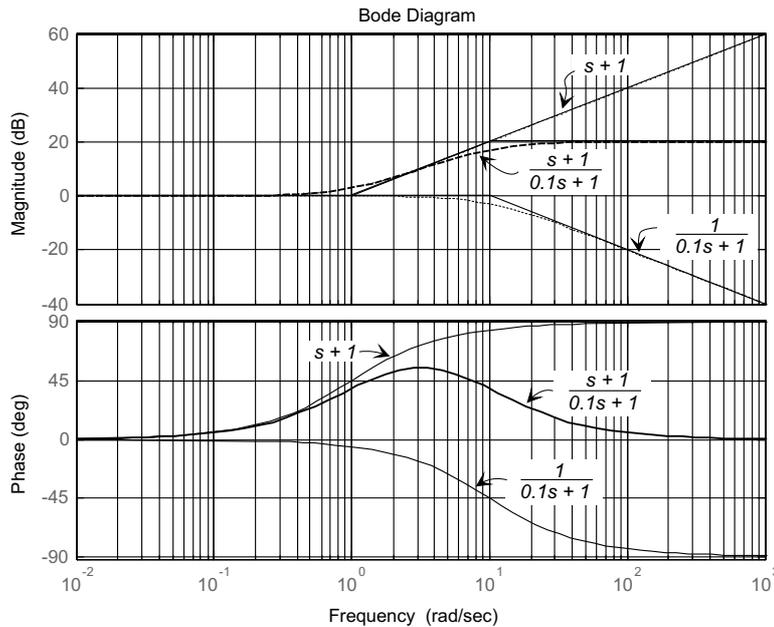
◼ Example 2

Plot the Bode magnitude and phase plots of a third-order system described by the transfer function

$$H(s) = \frac{40s + 4}{s^3 + 2s^2 + 2s}$$

Solution: The transfer function is rewritten

$$H(s) = \frac{4(10s + 1)}{s\left(s^2 + 2s + 2\right)} = 2\left(10s + 1\right)\left(\frac{1}{s}\right)\left(\frac{2}{s^2 + 2s + 2}\right)$$

indicating four component terms:

1. A constant gain term of $H_1(s) = 2$,

2. A single real pole at the origin $H_2(s) = 1/s$,

3. A complex conjugate pole pair $H(s) = 2/(s^2 + 2s + 2)$, characterized by $\omega_n = \sqrt{2}$ radians/sec. and a damping ratio of $\sqrt{2}/2$, and

4. A single real zero term $H_3(s) = (10s + 1)$, with a break frequency of $\omega = 0.1$ radians/sec.

The component terms are plotted and are added together to determine the total response for a frequency range of 0.01 to 100 radians/sec. in the magnitude and phase plots below.

The two Bode plots (magnitude and phase) show the individual responses $H_1(s) = 2$, $H_2(s) = 1/s$, $H_3(s) = 2/(s^2+2s+2)$, $H_4(s) = (10s+1)$, and the total response, plotted against angular frequency (rad/sec).
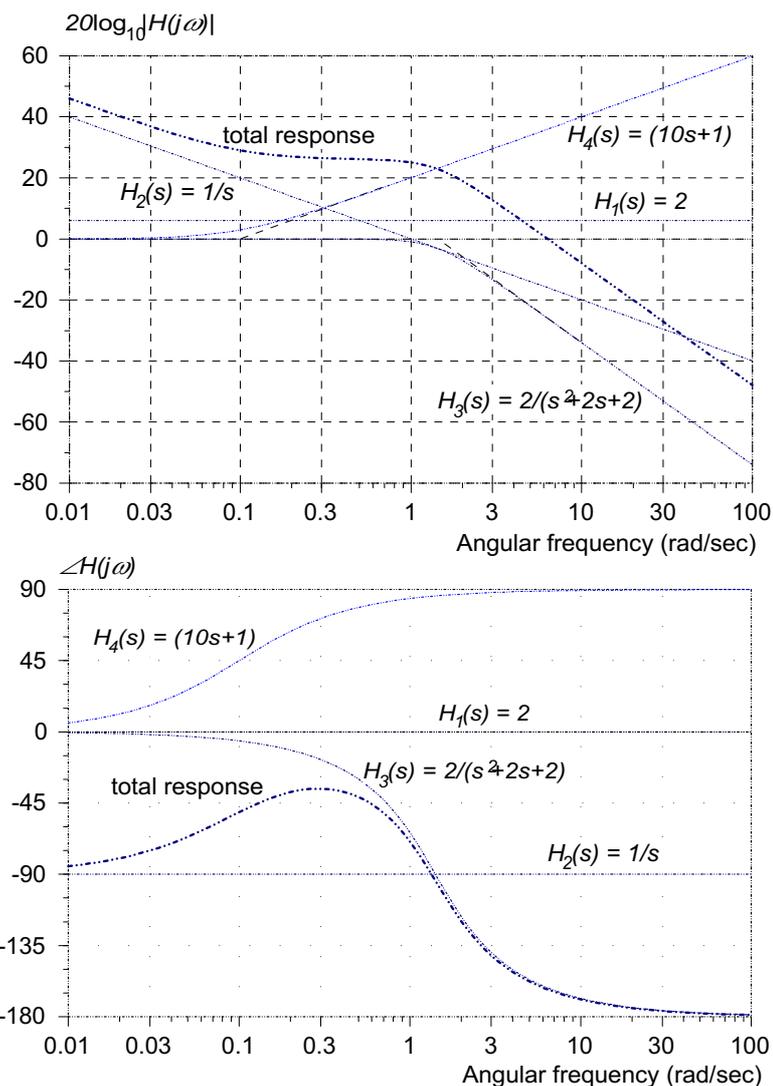
1.2 A Simple Method for constructing the Magnitude Bode Plot directly from the Pole-Zero Plot

The pole-zero plot of a system contains sufficient information to define the frequency response *except for an arbitrary gain constant.* It is often sufficient to know the shape of the magnitude Bode plot without knowing the absolute gain. The method described here allows the magnitude plot to be sketched by inspection, without drawing the individual component curves. The method is based on the fact that the overall magnitude curve undergoes a *change* in slope at each break frequency.

The first step is to identify the break frequencies, either by factoring the transfer function or directly from the pole-zero plot. Consider a typical pole-zero plot of a linear system as shown in (a) in the figure below. The break frequencies for the four first and second-order blocks are all at a frequency equal to the radial distance of the poles or zeros from the origin

of the s-plane, that is $\omega_b = \sqrt{\sigma^2 + \omega^2}$. Therefore all break frequencies may be found by taking a compass and drawing an arc from each pole or zero to the positive imaginary axis. These break frequencies may be transferred directly to the logarithmic frequency axis of the Bode plot.

(a)   (b)
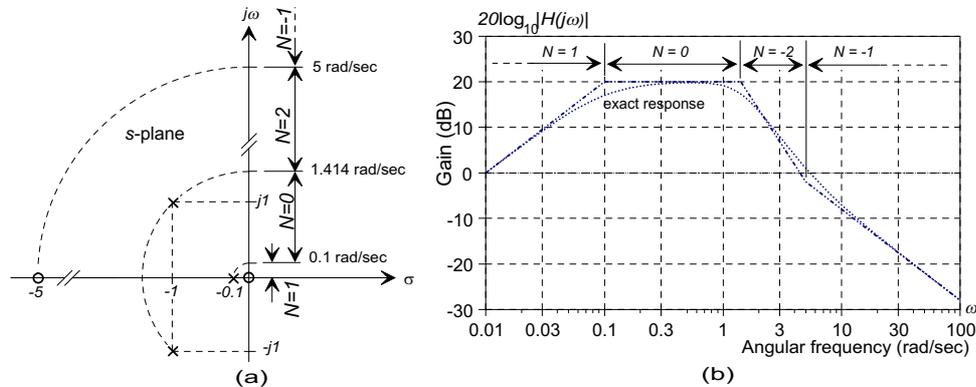
Because all low frequency asymptotes are horizontal lines with a gain of 0dB, a pole or zero does not contribute to the magnitude Bode plot below its break frequency. Each pole or zero contributes a change in the *slope* of the asymptotic plot of ± 20 dB/decade above its break frequency. A complex conjugate pole or zero pair defines *two* coincident breaks of ± 20 dB/decade (one from each member of the pair), giving a total change in the slope of ± 40 dB/decade. Therefore, at any frequency ω, the slope of the asymptotic magnitude function depends only on the number of break points at frequencies less than ω, or to the left on the Bode plot. If there are Z breakpoints due to zeros to the left, and P breakpoints due to poles, the slope of the curve at that frequency is $20 \times (Z - P)$ dB/decade.

Any poles or zeros at the origin cannot be plotted on the Bode plot, because they are effectively to the left of all finite break frequencies. However, they define the initial slope. If an arbitrary starting frequency and an assumed gain (for example 0dB) at that frequency are chosen, the shape of the magnitude plot may be easily constructed by noting the initial slope, and constructing the curve from straight line segments that change in slope by units of ± 20 dB/decade at the breakpoints. The arbitrary choice of the reference gain results in a vertical displacement of the curve.

In (b) the straight line magnitude plot for the system is shown, constructed using this method. A frequency range of 0.01 to 100 radians/sec was arbitrarily selected, and a gain of 0dB at 0.01 radians/sec was assigned as the reference level. The break frequencies at 0, 0.1, 1.414, and 5 radians/sec were transferred to the frequency axis from the pole-zero plot. The value of N at any frequency is $Z - P$, where Z is the number of zeros to the left, and P is the number of poles to the left. The curve was simply drawn by assigning the value of the slope in each of the frequency intervals and drawing connected lines.