2.004 Dynamics and Control II
Spring 2008

Massachusetts Institute of Technology

Department of Mechanical Engineering

2.004 *Dynamics and Control II*

Spring Term 2008

Lecture 18[1]

Reading:

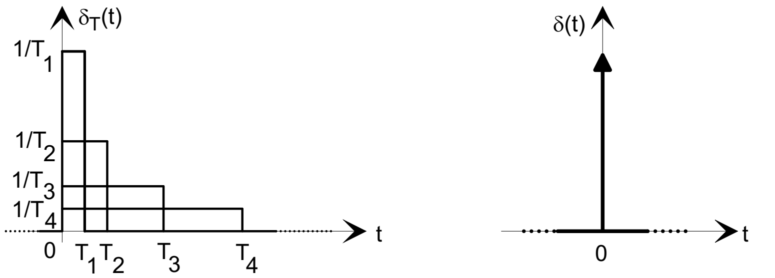
- Nise: Sec. 1.5

1 Common Inputs Used in Control System Design and Analysis

Two classes of inputs commonly used to characterize the performance of feedback control systems are:

(a) The "Singularity" Functions: These functions have a discontinuity at time $t = 0$:

(i) The Dirac Delta (Impulse) Function: The delta function is used to characterize the response of a system to brief, intense inputs. In the figure below, (a) shows some unit pulses (pulses with unit area so that if the duration of the pulse is T, its amplitude is $1/T$. The Dirac delta function $\delta(t)$ is the limiting case of such pulses as $T \to 0$. Notice that this implies that the amplitude $1/T \to \infty$.



a) Unit pulses of different extents   b) The impulse function

The strict definition of $\delta(t)$ is

$$\delta(t) = 0, \quad \text{for } t \neq 0$$
$$\delta(t) \;\; \text{is undefined at } t = 0 \quad (\lim_{t \to 0} \delta(t) = \infty)$$
$$\int_{0^-}^{0^+} \delta(t) dt = 1$$

(ii) The Unit-Step Function $u_s(t)$: The unit-step (or Heaviside) function is used to characterize a system's transient response to a sudden change.

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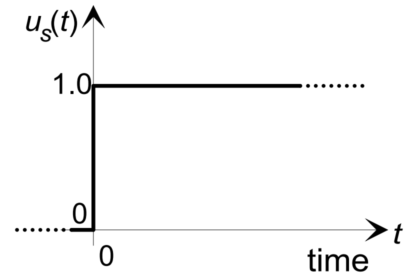
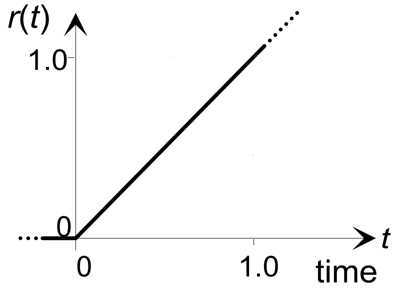


The definition is

$$
\begin{aligned}
u_s(t) &= 0 \quad \text{for } t < 0 \\
&= 1 \quad \text{for } t > 0
\end{aligned}
$$

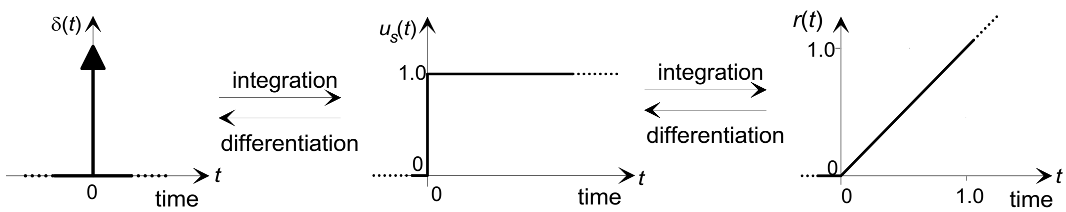
(iii) The Unit-Ramp Function $r(t)$: The unit-ramp function is used to characterize a system's ability to follow a time-varying input, and the transient behavior around a discontinuity in the slope of an input function.



The definition is

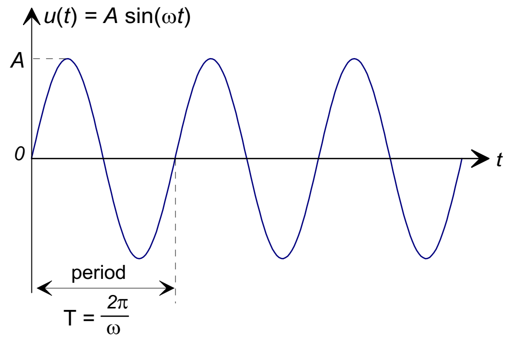
$$
\begin{aligned}
r(t) &= 0 \quad \text{for } t < 0 \\
&= t \quad \text{for } t > 0
\end{aligned}
$$

The singularity functions a related to each other by differentiation and integration, as shown below:



(b) Sinusoidal Inputs: The response of linear systems to sinusoidal inputs of the form

$$u(t) = A\sin(\omega t + \theta)$$

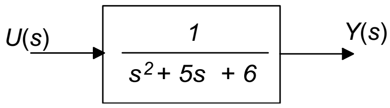


is of fundamental importance to control engineering and system dynamics ant will be studied extensively throughout the course.

■ Example 1

In practice a machine, described a second-order transfer function $G(s)$, will be subjected to inputs that change suddenly. Use the unit-step response to determine how long it will take the machine's response to settle to a new steady-state value after a change.

The system's block diagram is



and the governing differential equation is

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = u(t).$$

The step response assumes that the system is "at rest" at time $t = 0$, that is $y(0) = 0$ and $\dot{y}(0) = 0$, and that the input $u(t) = u_s(t)$.

The solution is

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ is the *homogeneous solution*, and $y_p(t)$ is the *particular integral.* The characteristic equation is

$$\lambda^2 + 5\lambda + 6 = (\lambda + 3)(\lambda + 2) = 0$$

and

$$y_h(t) = C_1e^{-3t} + C_2e^{-2t}.$$

Assume $y_p(t) = K$ and substitute into the differential equation

$$0 + 0 + 6K = 1$$

or $y_p(t) = 1/6$. The complete solution is

$$y(t) = C_1 e^{-3t} + C_2 e^{-2t} + 1/6.$$

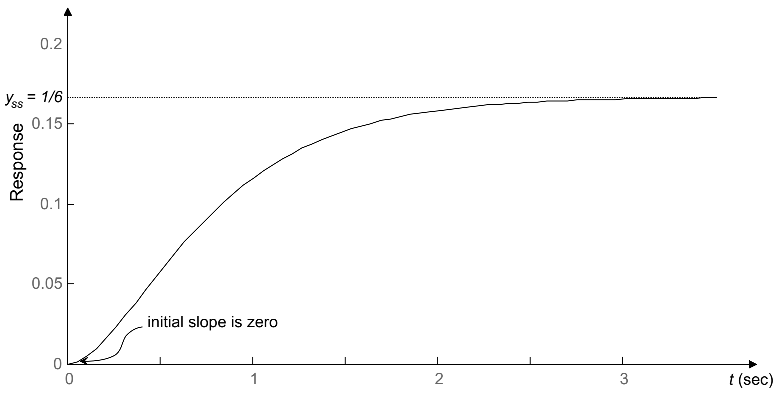
At time $t = 0$

$$\begin{aligned} y(0) &= C_1 + C_2 + 1/6 = 0 \\ \dot{y}(0) &= -3C_1 - 2C_2 + 0 = 0 \end{aligned}$$

giving $C_1 = 1/3$ and $C_2 = -1/2$, so that the system's response to a unit-step input is

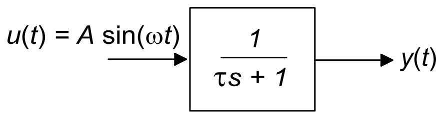
$$y_{step}(t) = \frac{1}{3}e^{-3t} - \frac{1}{2}e^{-2t} + 1/6.$$

The response is shown below, and indicates that it takes this system 2.5–3 seconds to respond to the step.



■ Example 2

Find the steady-state response of a first-order system to a sinusoidal input $u(t) = A\sin(\omega t)$.



The differential equation is

$$\tau \frac{dy}{dt} + y = u(t)$$

and assume the complete solution is $y(t) = y_h(t) + y_p(t)$ as in the previous example. The characteristic equation is

$$\tau\lambda + 1 = 0$$

from which

$$y_h(t) = Ce^{-t/\tau}$$

and assume

$$y_p(t) = K_1\cos(\omega t) + K_2\sin(\omega t)$$

In steady-state we assume that $y_h(t) = C_1e^{-t/\tau}$ has decayed to zero, and

$$y_{ss}(t) = y_p(t) = K_1\cos(\omega t) + K_2\sin(\omega t)$$

Substitution of $y_p(t)$ into the differential equation gives

$$\tau\omega\left(-K_1\sin(\omega t) + K_2\cos(\omega t)\right) + \left(K_1\cos(\omega t) + K_2\sin(\omega t)\right) = A\sin(\omega t)$$

or

$$(\omega\tau K_2 + K_1)\cos(\omega t) + (-\omega\tau K_1 + K_2)\sin(\omega t)$$

and comparing coefficients

$$\begin{aligned} \omega\tau K_2 + K_1 &= 0 \\ -\omega\tau K_1 + K_2 &= A, \end{aligned}$$

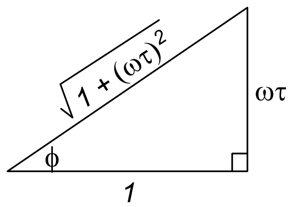
or

$$K_1 = \frac{\omega\tau A}{1+(\omega\tau)^2}, \qquad K_2 = \frac{A}{1+(\omega\tau)^2}$$

so that

$$\begin{aligned} y_{ss}(t) &= y_p(t) = \frac{A}{1+(\omega\tau)^2}\left(\sin(\omega t) - \omega\tau\cos(\omega t)\right) \\ &= \frac{A}{\sqrt{1+(\omega\tau)^2}}\left(\frac{1}{\sqrt{1+(\omega\tau)^2}}\sin(\omega t) - \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}\cos(\omega t)\right) \\ &= \frac{A}{\sqrt{1+(\omega\tau)^2}}\left(\cos(\phi)\sin(\omega t) - \sin(\phi)\cos(\omega t\phi)\right). \end{aligned}$$

using the following triangle



$$\sin(\phi) = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}$$

$$\cos(\phi) = \frac{1}{\sqrt{1+(\omega\tau)^2}}$$

Then the system sinusoidal response is

$$\boxed{y_{ss}(t) = \frac{A}{\sqrt{1+(\omega\tau)^2}}\sin(\omega t - \phi)}$$

where $\phi = \tan^{-1}(\omega\tau)$. We note the following

(a) The steady-state sinusoidal response is a sinusoid of the same frequency as the input.

(b) There is a phase shift (lag) between the input and output $\phi = \tan^{-1}(\omega\tau)$.

(c) The amplitude of the output is a function of the input frequency ω.

(d)
- At low frequencies ($\omega \to 0$), the response is $y_{ss}(t) \approx A\sin(\omega t)$ and the amplitude approaches that of the input.
- At very high frequencies ($\omega \to \infty$), the response is $y_{ss}(t) \approx (A/\omega\tau)\sin(\omega t - \pi/2)$ and the response amplitude becomes very small.
- When $\omega = 1/\tau$, $y_{ss}(t) = (A/\sqrt{2})\sin(\omega t - \pi/4)$, that is the amplitude is reduced by a factor of 0.707, and the phase shift is 45°.
