PROFESSOR: So what's important the last couple of lectures? Christina.

AUDIENCE: Transfer functions.

PROFESSOR: Transfer functions, all right. How about something else?

AUDIENCE: Modal coordinates.

PROFESSOR: Modal coordinates. And I'll expand that to call this modal analysis in general. Anything else? Well, think more about that. And are there any questions for the week, anything muddy, fuzzy, not quite clear to you that if I have time to say a few words about today you'd like me to talk about?

AUDIENCE: So I guess I'm a little bit confused with what we did in class, the modal analysis. Is that just like-- so when we did originally Newton's law, and then we did the [INAUDIBLE]. Is that sort of similar here where we can do either the things in motion, or we can do [INAUDIBLE] analysis?

PROFESSOR: Is modal analysis an alternative way of--

AUDIENCE: [INAUDIBLE]

PROFESSOR: The analogy isn't too helpful. But modal analysis is one way of attacking the equations of motion that describe vibration of linear systems. You can work through the entire analysis and figure out the solution to the equations of motion. And usually with these multiple degree of freedom vibration problems, you can cast them as a mass matrix times an acceleration vector plus a stiffness matrix times a displacement vector equals some forcing function.

So these are-- and we linearize them. And we know they're for vibration problems.

1

We can solve these equations using modal analysis. Or we can solve them brute force directly without breaking the results into their modal contributions. And so there really are two kinds of approaches you can use to doing it. Any other kind of questions? Yeah.

AUDIENCE: So for a definition of mode shape, is mode shape just the ratio of the two amplitudes?

PROFESSOR: In the case of a two degree of freedom system, so mode shapes for mode n of an n degree of freedom system-- like let's say n equals 3 here. So for mode n, the mode shape you always have to present some normalization. I do the normalization oftentimes by just saying, OK, I'm going to make the top coordinate-- I'm going to give it unit amplitude.

So I'm going to take a1 and divide it by itself. And that'll give me a 1. Then every other one, this becomes a2 over a1, a3 over a1. The mode shape says if this is then 1 minus 1/2 and a 1/4, it says that if generalized coordinate 1 moves a unit amount, generalized coordinate 2 will move half of that in its negative direction. This could be an x, a displacement, and that could be a rotation. But they all have positively defined directions.

So if there's a positive 1 at x1, there's a minus 1/2 at generalized coordinate 2, and there's a plus 1/4 at generalized coordinate 3. And that ratio stays constant for the mode. And so if it's an initial condition problem, and you set it up so that you actually give it an exhibit, just an initial displacement, of exactly that shape, it'll sit there and vibrate only in that mode.

And you'll notice that proportion, even as it dies out with damping, of the first to the other two stays exactly constant. That's what a mode is. And it's a character. It's a property of the system.

The natural frequency is a property. And the shape of every mode is a property of that system. OK, anything else, questions? Next lecture, we'll do more modal analysis. We did the response to initial conditions yesterday. Tuesday, we'll talk

about the response to external forces. And that will give us a chance to review it and post it on Stellar.

Last night, I put up a little two page sheet that is a cookbook, how to do the procedure of a modal analysis. It's very cookbook, just step by step, bang, bang, bang. And everything falls out. Any other questions, yeah?

AUDIENCE: So is that where we get the mode shape from?

PROFESSOR: You get the mode shape by solving the characteristic-- if you're doing it by hand, by solving the characteristic equation, which we're going to get some practice at today. So I think this is the exercise of today, is how to get natural frequencies and mode shapes.

AUDIENCE: Then you'd be actually controlling the amount that-- isn't the mode shape actually drawing some sort of graph?

PROFESSOR: As a graph that shows the displacement of it, or it shows the time history?

AUDIENCE: Either, I guess. Because in the book, they have mode shape. But actually using this, they draw a certain graph.

PROFESSOR: Sometimes they do. If it's easy, like for a vibrating string, it's easy to draw the mode shape.

AUDIENCE: And the piece that they ask us to do that.

PROFESSOR: Yeah, and so vibrating string, first mode vibration looks like half a sign wave. The mode shape is just half the sine wave going up and down in space. For this system, the mode shape has two different amplitudes at these two bobs. And their relative motion we're going to figure that out today.

OK, we better get going or we won't finish. So good questions. And here's this system simplified, just into two bobs and massless strings. Here's the equations of motion for that. They've been linearized already. There's no sine thetas or anything.

3

And part of the linearization-- so this assumes, then, small amplitudes, right? Not only did the sine-- the torque for a pendulum, you get this mgL sine theta term. And so the linearization, you say small angles, sine theta becomes theta. So that's where this g/L sine theta has become theta term.

There's another assumption that's been made to model this system. We're essentially modeling the system. And that is for small angles, we're assuming the spring remains horizontal. Because the spring puts a force on the rod.

And what you want is torque about that point. These two equations are the equations for torque about that point and about that point. And the torque is the spring force times that distance. And if they're perpendicular, that's just one times the other.

But if it doesn't remain perpendicular, then you'd have to take a cosine. And it gets really messy. So another part of this small angle approximation is that spring stays horizontal. There's your equation of motion.

And you can see what's happened here is that this used to have an m1 L1 squared. This is the mass moment of inertia of that mass. We've divided through the equation by m1 L1 squared. And that's put things in the bottom here like that.

And it just makes this actually a little easier to work with. The mass matrix turns out to be 1, 1 when you do this. But it's still the same two equations of motion. I've just divided through by m1 L squared, and this one by m2 L squared.

So these are your equations of motion. And your job is to come up-- first of all, just put them in matrix form. And you're going to do this in groups. Today we've got five, 10, and 18 people. So that's probably four groups of four. Work together in groups quickly. Let's put up the equations of motion from this in matrix form.

AUDIENCE: [INAUDIBLE]

PROFESSOR: In your lower left matrix? Is the-- oh, the k. I'm worried about the m.

AUDIENCE: The k's don't have any symbols. it's just m2.

PROFESSOR: There you go. Yeah, it's just one. There's a single k. There's only one spring, so we just call it k. All right, you're all looking pretty good to me. And the last hour, we did this, and it went up, and I looked at it, and I just had one of these moments of just, uhh, cognitive dissonance. It just was, how can that possibly be?

And the reason was this stiffness matrix. This matrix is not diagonal-- excuse me, wrong word. This matrix is not symmetric. And linear stiffness matrices are always symmetric.

So I saw this, and I said, what has gone wrong? Every one of you had the same answer. Every one of you, your stiffness matrices are not symmetric. It's minus k over m1 and minus k over m2.

And I about had heart failure, and I've been doing this for 40 years. And it's always been symmetric. And all of a sudden, you guys unanimously get asymmetric. So I figured it out at great relief.

We divided through by m1 L1 squared in the first equation, and m2 L2 squared in the second equation. The stiffness matrix, the true stiffness matrix, hasn't been divided through by the mL squareds. And it is indeed symmetric.

It's one of the things that's really helpful to know. Because when you're working these things out, you know they're symmetric. You don't have to evaluate all the terms. You only have to do the diagonals and half of the ones.

The mass matrices are usually symmetric, too. There's maybe some exceptions if you start putting in gyroscopes and stuff like that. But then it's not very linear. So your next task is to-- the way we find natural frequencies and mode shapes is kind of the hard way, using algebra, is to find what we call the characteristic equation.

It turns out to be a fourth order equation, omega to the fourth. Find it. Don't solve it, just write down the characteristic equation for this thing. And as soon as you have that, come up and write it down.

So remember, you basically have matrices. This is what you're working with. You've got something of this form. We've divided through by some stuff to collect terms. But how do you go about going from the equations of motion to your algebraic equation in omega to the fourth?

I just want you to write up the characteristic equation. You're going to get a quadratic in omega to the fourth. Just go up and write down the equation.

AUDIENCE: So do you want us to expand?

PROFESSOR: By expand, what do you mean?

AUDIENCE: [INAUDIBLE] multiply this by this? But we could also expand that out.

PROFESSOR: Well, I want an equation in omega to the fourth. Just write one up there. And if you want to simplify the writing, you can let h equal-- and let's see, I forgot the tell you something, a key assumption here, a key thing. I was going to make it easier for you. Let--

AUDIENCE: Are you going to say the m's are equal?

PROFESSOR: Yeah, sorry, my mistake. Let the m's be equal. Then it makes it a lot easier. And then you can say, call h g/L plus k/m. And it'll kind of make the equation a whole lot easier to write.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Great, OK, people are getting all the same answers here, good. So the next step is I will give you-- if you solve this, you get omega 1 is g/L. Omega 1 squared is g/L. And omega 2 squared is g/L plus 2k over m.

Those are your two natural frequencies. So now, find the mode shapes. Find the mode shapes. If you know the natural frequencies, now you go back in and you get a-- pardon? Yeah, keep h. I think h will-- things will fall out rather quickly.

Well, I like to normalize the mode shapes so the top one is 1. And so the top

element of the mode shape vector I call a1, or u1, down through un. And so if you divide out each one by the top one, the top one becomes 1. The second one becomes a2 over a1. The third one, if it's three degrees of freedom, would be a3 over a1.

The fact that this-- you just solve this equation. To get that characteristic determinant, you said this was true. And that's a particular kind of situation where you have a set of linear homogeneous equations equal to 0. They're not independent. Remember from algebra when you have little equations with constant coefficients, and they're equal to a constant, and the constants are all 0, means they are not independent equations.

So this is two equations and two unknowns. And they're not independent. You won't be able to solve for unique values of a1, u1, and u2. You'd only be able to get the ratio.

So basically you found this value. You're going to plug in a value for omega squared and find out the values of a1 and a2 that work. That's what the mode shapes are. Let me pull you back together here, and let's run through this kind of quickly.

You've basically solved this equation. You've said, I'm going to assume this thing has a mode shape and a frequency, a harmonic-- it's going to vibrate. It's undamped. So either you could write this as cosine omega t, sine omega t, e to the i omega t. But it has some constants out here called the shape.

You plug that into the equations of motion, you're going to get back this expression. The two derivatives, the theta double dots, give you the minus omega squareds. And you can write it like that. You can throw away the e to the i omega t.

In this case, I've expand this. m is just that 1, 1 matrix. So this part looks like this. The k matrix looks like this. You can add them element by element. So this is minus omega squared plus h, minus k/m. This one is minus k/m, and this term is minus omega squared plus h again times u1 or u2, the two elements that we're looking for.

That's two equations. This is just now an algebraic equation, two algebraic equations-- this time u1 minus k/m times u2. And the second equation is that. And we know there's only two equations, and not linearly independent. So you actually only have one useful equation. If this is a three by three, they're not independent. But you need to use two out of the three. Yeah.

AUDIENCE: How do you get the second equation?

PROFESSOR: The second one, I take the first one, and I multiply it up here. And that gives me an equation, this times u1 plus this times u2. That's equation one. I take this, and I multiply it by that. And I get minus k/m u2. And this is u1. This is the omega squared. I plugged in the natural frequency. So we solved for it.

AUDIENCE: Oh, OK, OK. But for the top one, you didn't plug in the natural frequency.

PROFESSOR: Oh, minus omega squared. OK, I'm picking one of the natural frequencies, g/L. So this up here is g/L plus h u1, that one. And down here it's g/L h u2. And you see, these two equations turn out to be--

AUDIENCE: So this is just for omega 1?

PROFESSOR: Yeah, you only do it one at a time. You just plug in one of the natural frequencies. And you get two equations and two unknowns. And they're going to each give you exactly the same information. They're not independent any longer.

So you solve this one. h has what in it? A plus g/L and a plus k/m. The g/L is canceled. And you're left with k/m u1 minus k/m u2 equals 0. And this implies that u1 equals u2.

This is the g/L in this. I'm going to plug in the-- that is h. I just put it in here. I've plugged in one of the natural frequencies squared. The g/L parts cancel. This is plus k/m u1 minus k/m u2. That means k/m u1 equals k/m u2. Cancel out the k/m's. I get that. It just tells me that that's the answer.

And if that's the answer, the vector looks like u1, u2. That's also equal to-- factor out a u1-- u1 times 1 and u2 over u1. That's the mode shape there, 1 and u2 over u1.

That's how you factor it out to put it in the normalized form. You just take whatever's in the top one and divide everything in the column by it.

So to do the second equation, you just now go back to this, plug in for omega 2 squared. The second natural frequency is g/L plus 2k over m. This is g/L minus k/m. The k over m's cancel. And you're just left with g/L's. You work this through, you're going to find out that u2 equals minus u1 if you put in the second one. Yeah.

AUDIENCE: So we have a1 over a2, b1 over b2, being equal to 1, obviously. Because b1 is equal to b2. But I don't understand the format that you're writing it up there. Because if you plug back in the u, then you just get u1 over u2 is equal to u1 over u2. And that's self-explanatory, right?

PROFESSOR: Well, when you solve this, you find that k/m u1 equals k/m u2. That's just saying in this case they are equal, period. And that's all you learn. You only have two equations. And they're not independent. So it means you only have one equation. And you don't learn-- you're not able to solve for numeric values of u1 and u2. At best, you can get u1 over u2.

AUDIENCE: Yeah, because they're equal to [INAUDIBLE].

PROFESSOR: Right, and in fact, you can just solve this for u2 over u1, and it would be 1.

AUDIENCE: I just don't understand the format you're writing it in up there.

PROFESSOR: Well, that's back in my vector format. I want to write the mode shape as a vector. You solve for u1. So you can actually do this. You know that that's true. You could say, well, the top one is 1, and the bottom one is the same as that. So the mode shape for mode one is that.

AUDIENCE: In the answers, it says omega 1, and the ratio of a1 and a2 is 1. What are the a's?

PROFESSOR: I'm using u's. And they used a's. They're the elements of the mode shape vector. So if we had put in the other natural frequency instead of-- I may have just done too many steps here at once.

So our first equation looks like minus omega squared plus h minus k/m. And the second equation is minus k/m. And over here is minus omega squared plus h again. That's our matrix. When we add these two matrices together, the minus omega squared m plus k, that's what it looks like when you add elements together.

And these are two equations and two unknowns. We're looking for the answer for some u1, u2 multiplied by this. So you multiply this out. You get minus omega squared plus h u1 minus k/m u2. And now let's plug in the second natural frequency. So omega 2 squared is that.

And solve for the second mode now. Plug it in here. We get minus g/L minus 2k over m plus h, which is g/L, plus k/m plus, and that's times u1, plus k/m u2. All it's equal to 0. This cancels this, this, and this, plus 1 minus 2. I get a minus k/m out of this. So I have this whole thing gives me minus k/m u1 plus k/m u2.

AUDIENCE: It's supposed to be minus k/m.

PROFESSOR: I agree with you. Where did I make the mistake?

AUDIENCE: The very first line, it's minus k/m u2.

PROFESSOR: Ah, good, all right. And this is equal to 0. The k over m's cancel out. And you find that this implies that u1 equals minus u2. And so the mode shape for mode two you could write as u1. And u2 is minus u1, if you will. And it's obvious if you factor out u1 you get u1 and a 1 minus 1, right?

So now you've solved for the second mode shape. If this had been more than a two by two, like a three degree of freedom system, you'd have to use two of the three equations to get the ratio. You'd get two equations and three unknowns. And at best, you can find the two of them in terms of the third. And the third one would be u1. You'd just divide all the others by u1. So you only get their ratios.

OK, so those are the natural frequencies. Those are the mode shapes. Let's demonstrate it. Let's look at it. This is basically the system. And you now know the mode shapes, the two mode shapes of this thing, are 1, 1 for the first mode and 1,

minus 1 for the second mode.

And the modal expansion theorem depends on this fact that the mode shapes form a complete independent set of vectors that are orthogonal to one another. A weighted sum of all the mode shades of the system can represent any allowable motion of the system. Any possible motion of the system can be written as a weight of-- so any way I can displace these things, a little bit here, a little bit there, I can write that position as a weighted sum of those two mode shapes.

So let's say I want to have initial conditions on displacement, on r theta, 1 0 and theta 2 0, no initial velocities, just initial angles. And I want that initial condition vector to be 1, 0. I'm claiming I can write that as a sum of something 1, 1 plus another something 1, minus 1.

So what do c1 and c2 have to be to give me that? You ought to be able to kind of do that by inspection, almost. For example, just try them equal. And what happens?

AUDIENCE: [INAUDIBLE]

PROFESSOR: So 1/2 and 1/2, right? Great, so to satisfy this, it's 1/2 1, 1 plus 1/2 1, minus 1. And I've just illustrated the fact that you can represent an arbitrary deflection, allowable deflection of the system, as a weighted sum of the two mode shapes.

So that says if I make the initial-- now, this says that theta 1 is 1, and theta 2 is 0. And what it's telling us is by modal analysis, that means the resulting motion should look like the response to initial conditions where I have equal amounts of each of those two modes. So there ought to be some vibration at omega 1, and there ought to be equal amount of vibration at omega 2. Agreed?

OK, let's try it. I'll hold one of these in place, and I'll deflect the other one and let go. Now, that one's stationary, and this one's moving. And a few cycles later, this one will be stationary, and that one will be moving.

So what you're observing, could that motion, what you're seeing, possibly be a mode, a natural mode by itself? No, because the different proportions are changing,

right? This is the sum of two modes-- exactly as you said, equal amounts of each.

And when you do that, and you have two things of equal amplitude, two cosine omega t like terms of equal amplitude and different frequencies, you get a phenomenon known as beating. That's what this is.

Beating means that one vibrates, and then it will get small. And then you'll see it build up again, and then get small. So if you watch either one of these, it's doing large and stopping. And the other one is doing the same thing, but actually 90 degrees out of phase.

So what we're really seeing is that the motion of this system, this is 1, 1, 1/2 and 1, 1 cosine omega 1 t plus a 1/2 1, minus 1 cosine omega 2 t. That's the total response of the system. Let's look at the first line of this. This says 1/2 cosine omega 1 t plus 1/2 cosine omega 2 t should be equal to the motion we call theta 1 of t. That's the first row here.

This is theta 1, theta 2. So the first row of this, the first equation, is that. And two equal amplitude cosines of different frequencies you can write as cosine omega 2 minus omega 1 over 2 times t times cosine omega 1 plus omega 2 over 2 t.

This beat phenomena that you're seeing when you plot this-- and this is now for theta 1. It starts off at some amplitude. And the actual motion you see, if you watch it, it does what I'm drawing right now.

This envelope is this term, the difference frequency divided by 2. What's inside is that term. And this is called beating. This is the equation for beating. When you add two equal amplitude cosines together, they give you something that looks like that. And one period of the beat is how long it takes to go through one full cycle from here to here. That's one period of the beat. Yeah.

AUDIENCE: How did you get those terms with the omega 2 [INAUDIBLE]?

PROFESSOR: Well, that's just trig identity. You add-- you just go back to your trig, take cosine of a plus cosine of b. You'll find out you can do that. I don't have time to go through that.

And the other one is a 1/2 cosine omega 1 t minus 1/2 cosine omega 2 t. And that turns out to be sine omega 2 minus omega 1 over 2 t times sine omega 1 plus omega 2 over 2 quantity times time.

And that means that the theta 2 coordinate, what it looks like when you draw it, is 90 degrees out of phase. It's the same thing, but it starts at 0 here and beats. But it starts 0 where that one started here. And that's what-- if you look at one.

So if that's theta 1 and this is theta 2, when one of them stops, that's one of these lines, one of these equations. And the other equation is 90 degrees. So that one's stopped right now. Pi over 2 later, this one will be stopped. So the beat behaves like sine in one case, and behaves like cosine in the other case.

One of these needs a minus sign. That's a plus, minus and plus, minus and plus. Now what if you have unequal amounts? We have exactly equal amounts of the two modes. If you have unequal amounts of the two modes, then it's not going to be 1/2 and 1/2. What if it's like 1 and 0.1? Then do you see full beats? Does one come to a stop?

If they're not equal, you'll find that the sums of the two motions, the envelope will look like this. It'll be modulated. Each one's motion will look like that. It'll never go to 0 quite. But that's beating. How are we doing on time? We're good.

And let's see, we said the one natural mode is the two opposite one another. So if I start, go equal amounts in opposite directions and let go, it ought to just sit there and vibrate all day like that-- no beats. Because it's the 1, 0 case over there.

And actually, this is the 0, 1 case. This is 1, minus 1. I gave an initial displacement in exactly the shape of the second mode. And that means the two contributions are that one of those constants is 0. There's no mode one in this. It's only mode two. And it'll sit here all day long and vibrate just in mode two.

And if I displace it in the shape of mode one only, it'll vibrate. And that one's harder do. Because I have to move it exactly the same amount. That's mode one. And it should vibrate all day long on mode one. Because now there's no mode two

involved in that one.

You see it's already getting a little out of phase. It's hard for me to move my hands exactly the same amount. So I have a little bit of the other mode in there. But you'll never see either one of these come to a full stop. It's actually doing that when you get a little contamination of the second mode in there.

OK, so I've posted on Stellar a little two-page sheet that just gives you step by step how to do a modal analysis. It's just cookbook. Modal analysis is easy. You do it all on the computer. You just put your m matrix and your k matrix, and you just crank stuff out.

We did multimodal analysis yesterday just from initial conditions. Next Tuesday, we'll do modal analysis assuming you've got harmonic excitation and steady state vibration and do that kind of thing.