PROFESSOR: Today we're going to talk all about moments of inertia, and the last time we did [? muddy ?] cards, there were a lot of questions. The most common question-- where did all those terms come from? And I'm going to do a brief kind of intro around that so as to facilitate the rest of the discussions.

Some basic assumptions here today. We're going to consider bodies that rotate about their center of mass, or fixed points. That's the kind of problems we're doing. And secondly-- all through this lecture today, there's four or five significant assumptions or conditions that are really important to the whole discussion. I'm going to really try to highlight them. And starting with two here. The other one is we're going to use reference frames attached to the bodies.

That's why we did all that work in kinematics figuring out how to do velocities of things that are in translating and rotating references frame, it's so that we can do these problems. So this said, here's a inertial frame. Here's some rigid body. Here's a point on it, that's my origin. And here is a set of coordinates attached to the rigid body-- an xyz system-- the body rotates and translates, it goes with it.

And if we consider a little piece of this body, a little part-- I'll call it little mass element mi-- from here to here is some distance y with respect to A, then the angular momentum of that little mass particle is by definition of angular momentum Ri cross Pi. And the P is always the linear momentum defined in an inertial frame. That's the definition of angular momentum.

So you have seen this little demo before. So this is now my rigid body that I'm talking about. In this case, the axis of rotation is this one. And I'm just going to define my rigid body as essentially rotating about this fixed point, this is A now. And my coordinate system is xyz, so the rotation is omega z. And this rigid body-- this is massless, consists of a single point mass.

So we've done this before, but we tended to do this kind of problem using polar coordinates. And now I really wanted to strictly think in terms of XYZ. The answers come out exactly the same. Polar coordinates also moved. You could do this in rhat, thetahat, z. We're going to do it in xyz attached to the body. And y is going into the board, and we'll have no values in the y direction for this problem.

Here's our little mass, we'll call it m1. It's at a position out here x1, z1 are its coordinates, and y is zero. And we want to compute its angular momentum, h1 with respect to A. That's our R1A cross P1 and 0. And for this problem that's x1 in the i hat plus z1 in the k hat, that's the R-- cross P and P is mass times velocity, m1. And the velocity is just omega cross R. We've done this before. So we need a length on this thing.

We've done this enough times before I'm assuming you can make this leap with me. The momentum is into the board, in the y direction. The linear momentum is R cross with that. And that will give us-- let me write it-- omega z. So that's the linear momentum. It's in the j direction, its velocity is x1 omega z. The radius times the rotation rate is the velocity, the direction is that direction. So this is your momentum.

You multiply out this cross product, you get two terms-- m1 x1 squared omega z k, minus m1 x1 z1 omega z i. And these two terms then we would identify. This is the angular momentum in the k direction, angular momentum in the i direction. There doesn't happen to be any in the z direction, in general there could be. But this is h for this particle, particle one. I want to particularly emphasize this one-- oops, this is-- excuse me-- z.

And this is the piece we call hx. So there's a complement of the angular momentum that's in the z direction, there's a component in the x direction. We know that if we compute-- just to tie it back to previous work a little further-- d by dt of h1A gives us the external torques in the system, a vector of torques when we compute that out.

I'm not going to calculate it, I don't need it for the purposes of the rest of the discussion. This one gives you omega z dot term. This one gets two terms because you have to take the derivative of i. So you get three terms which are torque in the x, torque in the y, torque in the z directions. Two of them are static, because it's trying to bend this thing back and out, and one of them, the z direction one, is the one that makes it spin faster.

So we've seen that, that's a review. And to make the leap from that to rigid bodies-- Capital H, now, a collection of particles with respect to its the origin attached to the rigid body is going to be the summation over all the little mass particles. Their position vector crossed with the linear momentum of all those little mass particles. Sum all that out.

Because it's a rigid body, this is always something of the form that looks like omega cross R, m omega cross R. And I can then write this as the summation of [? oper ?] i of the mi, RiA cross omega with respect to 0, cross RiA. That gives you the velocity, m gives you the linear momentum, crossed with this one again makes it into an angular momentum. So the angular momentum of every particle-- the bits of this come out looking like an R squared omega. Or at least has dimensions of R squared omega.

These can be like x1 squared, but these can also end up being terms like x1 z1, those cross product terms. But in general, these things all add up. And this is a vector, this is a vector, so this is a vector-- these are all vectors, the result's a vector. And therefore we would break this down, this whole thing. Once you write all that out and sum it all up, you're going to get a piece of it that we would say that is in the i hat direction and we call that Hx.

Another piece that's in the j hat direction, which we call Hy and Hz in the k hat. That's just how that all shakes out. In general you get three parts. Yeah?

AUDIENCE: Why do you have mass multiplied by displacement and the momentum?

PROFESSOR: Why do we have the mass multiplied by a position vector times the momentum?

Because that is the definition of angular momentum. It's R cross P, all right?

[INTERPOSING VOICES]

PROFESSOR: Ah, I see what I've done. Yeah you're right. This m doesn't belong-- I got ahead of myself-- it doesn't belong here. It pops out of this to here. Thanks for catching that. Absolutely right. So every term here is made up of things that look like x1, z1, m1 omega x. Or omega y, or omega z, because this is a vector, and it can have three components-- a piece in the i, a piece in the j, a piece in the k direction.

So there's a lot of possible terms, and we're in the practice of writing this out. This H vector can be written then as a-- I'm going to run out of room here. So the Hx term-- the piece that comes out with all little bits in the i direction-- when you just break it apart, we would write it Ixx omega x, plus Ixy omega y, plus Ixz omega z. All the i terms that float out of this, we just collect them together-- all the ones that are multiplied by omega x ends up being some terms that look like sums of mi-- x, mi z squared terms, and there's some y squared terms.

But we collect them all together and we call this constant in front of it Ixx. Just what floats out of this stuff. We break it into three pieces, the part of the angular momentum due to the rotation in the x, the part due to the rotation in the y, the part due to the rotation in the z. And you do the same thing for Hy, and you get in Iyz omega x plus an Iyy omega y plus an Iyz omega z, and you're finally get and Hz term. Izx omega x, Izy omega y, Izz omega z.

That's everything that falls out of just doing this calculation. And where in the habit of writing that in a matrix notation as the product of this thing called the inertia matrix. And so forth, with this bottom term being Izz multiplied by omega x, omega y, omega z.

You multiply that times the top row, you get Hx, middle row, you get Hy, bottom row, you get the Hz. So where this stuff comes from is just from carrying out the summation over all the mass bits. So it basically starts with the definition of angular momentum. And this is just a convenient way to write it. So for example, the Ixz term-- the one in the upper right there that's part of Hx, the Ixz term-- is minus the summation of all the little mass bits times their location xi zi. And this, the xz, always matches xz.

And when you want to do this over a continuous object, you integrate that. And for Izz, this is the summation over i, all the bits of mi xi squared plus yi squared. So in general this, by summations, that's where these terms come from. They just come. They've started with this calculation. And in general, from that calculation. So for our one particle system, for this thing. So where this is x and z, it's one single particle. So for our one particle system.

And there's no y1 in here because it's a zero in this problem. The coordinate of that little mass particle is x1, 0, y1. So in general, this is x1 squared plus yi squared, but that's 0. So it's just mx1 squared. So that's where all these things come from. Now let's kind of look at-- the units of these things are always mass times length squared.

The diagonal terms the Ixx, Iyy, Izz terms will all be of the form of something squared. And what it is is it's always just the distance of the mass particles from the axis of rotation, call that r. It's just a summation of this perpendicular distance from the axis of rotation. For Izz, the axis of rotation is z. And this piece here is always the distance squared that the mass particle is from the axis of rotation.

Now that's the set up for today. And when we want to get onto the real meat of the discussion, talking about things like what principal axes are and so forth. Now we're going to move on to the part of this conversation about principal axes. So a really important point.

For every rigid body, even weird ones. For every rigid body, there is a coordinate system that you can attach to this body, an xyz set of orthogonal coordinates that you can fix in this body such that you can make this inertia matrix be diagonal. Any weird body at all, there is a set of coordinates that if you use that set, this matrix turns out to be diagonal. And what that means in dynamics terms is if you then rotate the object about one of those axes, it will be dynamically balanced.

And so when you rotate a shaft-- this is an object for which I know-- it's a circular uniform disk, one of the axes through the center is a principal axis. And if I rotate the object about that-- this one the hole's kind of out around so it wobbles a bit but if I rotate it about that it will make no torques about this axis that are because it's unbalanced.

It's essentially the dynamic meaning of principal axis-- an axis about which you can rotate the thing and it'll just be smooth, no away-from-the-axis torques. So that's so those are pretty important. We want to know what those are for rigid bodies and how to find them. So there's a mathematical way to find them and you can just read about in the textbook. It's sort of a-- I'm trying to think of the mathematical term. I forgot it. But what I want to teach you today is for many, many objects, if you just look at their symmetries, you can figure out by common sense where their principal axes are. So that's what we're going to do next.

We started off saying that we're talking about bodies that are either rotating about their centers of mass or about some other fixed point. So we're going to define our principal axes assuming we are rotating about the center of mass. There is an easy method known as the parallel axis theorem to get to any other point. So why do we care about doing it about the center of mass? Why is that useful to know about the properties around the center for dynamics purposes? What kind of dynamics problems do you care about the center of mass? Rotation about the center of mass.

AUDIENCE: [INAUDIBLE].

PROFESSOR: So when I throw this thing in the air, it's rotating, it's translating, and when it rotates, what's it's rotating about?

AUDIENCE: Center of mass.

PROFESSOR: Center of mass. So there's just lots and lots of problem in which in fact the rotation is going to occur around the center of mass. Any time the thing is off there and there's nothing constraining its rotational motion, the rotation will be about the center of mass. So that's a good enough reason, and a very practical reason then for computing these things or knowing how to find these things about the center of mass.

All right so we're now going to do principal axes, and we're going to teach you some symmetry rules. Maybe first a little common sense. This thing-- is this a principal axis? No way, right? And we know for a fact that it has off-diagonal terms because I've defined my coordinates as being an x, y, z, like this, and this is often at some strange angle. So as a practical matter, how could I alter this object so that this is a principal axis?

Essentially, how would you balance this?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Take the mass off, fix it.

[LAUGHTER]

Or? All right. Where do I put it? Down here? Ah, you want it like this. This is steel going into aluminum, I have to be careful that I don't strip the threads. There we go. And now test one, this is axis of rotation, is it dynamically in balance? Smooth as silk. This now is a principal axis.

Its mass distribution is symmetric. Where is there a plane of symmetry for this problem, for this object? That's what we're going to talk about now, planes of symmetry and axis of symmetry. Ah, you're saying one like back? You're right, that's a plane of symmetry. Is there another one for this one? Like this. So there's two planes of symmetry.

So let's say come up with some symmetry rules. So this first one is there exists-- by this I mean diagonal, it's a diagonal matrix. You can find a set of orthogonal axes such that this matrix is diagonal. That's what we mean when we go to find principal axes. And pick a board here-- rules. And we're going to have three of them, so just leave room on your paper. We're going to build these up. And I'm going to talk about it, and then you add another one and so forth.

So rule one.

AUDIENCE: [INAUDIBLE] the inertia matrix [INAUDIBLE].

PROFESSOR: What is the inertia matrix--

AUDIENCE: No, I mean the second one.

PROFESSOR: This thing with the diagonals? This is going to have all the off-diagonal terms are zeroes. That's what I mean. It's a diagonal matrix, the mass moments of inertia and products of inertia-- you only have Ixx, Iyy, Izz if you properly pick this set of orthogonal coordinates attached to the body-- if you pick them right.

What it means if then you spin that object about one of those axes, it's in balance. So rule one, if there it is an axis of symmetry-- Remember the caveat here is we're talking about uniform density objects or at least objects in which the density is symmetrically distributed. So the geometric symmetry and mass symmetry mean the same thing.

What does that mean? So axis of symmetry. Does this have an axis of symmetry? Where?

AUDIENCE: It has multiple ones down the middle and it's also symmetric [INAUDIBLE].

PROFESSOR: That's a plane of symmetry, the [? other ones you think, ?] axis of symmetry.

AUDIENCE: Well I mean the line right through it.

PROFESSOR: Which way?

AUDIENCE: Well any.

PROFESSOR: Nope.

AUDIENCE: Perpendicular to it.

PROFESSOR: Axis of symmetry means that there is a mirror image from that point-- that point mirror image over here is always identical. But you think any from angle, it doesn't matter where you start, it's reflected through the axis. So things that are circularly shaped tend to have axes of symmetry. So this has an axis of symmetry. Everything is reflected exactly just across the axes, you don't have to pick any particular line.

So what it's saying if it's an axis of symmetry, then it is a principal axis, and rotation about that axis-- the things will be nice perfectly in balance. If you go through all the hairy details of calculating the hard way, the Ixx, Iyy, et cetera terms-- all of the off diagonal terms will come out zero. And the reason is that for every mass particle over here, let's say there's a little bit right here in the corner, there is one exactly like it on the other side.

So if this one we're trying to bend this thing up as it spins around, this one over here is telling it to come back. Yeah?

AUDIENCE: How come this is not an axis, if you hold it-- yeah--

[INTERPOSING VOICES].

PROFESSOR: So if you do this. So I think it's kind of just the definition we're getting to of an axis of symmetry. Because in simple terms, what the object looks like this way is it's a narrow object, it is in fact a mirror image reflection. But it looks different when you go to this angle. And it looks different when you go to this angle so it's not an axis of symmetry, that's a plane of symmetry. We're going talk about that next.

So axes of symmetry-- the thing essentially looks the same at all orientations about that axis. But it is certainly dynamically balanced. Every mass bit here balances one over there, it doesn't wobble. All right. So that's for sure true. Now keep in mind that exists a set of orthogonal axes. So that means once you find one, you know that the other two are perpendicular to it and perpendicular to each other. So for this system, when I got the x and y here, there has to be an x and y perpendicular that that are the other two.

And because it's actually symmetric, it doesn't matter where you out them, you just could say it's these two, these two, it doesn't matter. You just have to decide where you're going to put them on the object, they're all the same. So for this axis symmetry, they're just two more. They're going to be in the body perpendicular to the first. Now we're going to do about g so the book uses g to talk about the center of mass, so I'll use g. So the center of mass of this thing is right in the middle of this and in the middle here.

So it's inside this body. Right in the dead center of it. So the three principal axes are an orthogonal set, one of which goes through it and the other two are embedded in it right angles. So this is the simplest rule. If you have an axis of symmetry, you know right way just about everything you need to know about the principal axes of the body. Shoot.

AUDIENCE: It seems like then there would only be axis of symmetry in objects that are either round or spherical, is that correct?

PROFESSOR: Pretty much.

AUDIENCE: So it's a very limited special case.

PROFESSOR: Yeah but there sure appear an awful lot in machines because they have this beautiful property of just being balanced in all directions. So everything rotating in the world tends to have an almost perfect axial symmetry. Just thinking about order here. Let's do this and then I'm going to go do a couple of examples.

The second rule-- if you have one plane of symmetry, a plane of symmetry. So this is kind of the opposite direction, in this you have the least information. Here's an object, does it have a plane of symmetry?

AUDIENCE: Yes.

PROFESSOR: Where?

AUDIENCE: Straight through [INAUDIBLE].

PROFESSOR: Show me. That cut. All right. So I've got these little dotted marks on this thing. So if I slice through that, I create two pieces that are identical. So that's a plane of symmetry. There is image match across that plane at every point. So if I have a plane of symmetry, what do you think you can say about a principal axis? One of the principal axis?

AUDIENCE: It'll be on that plane.

PROFESSOR: It'll be on that plane. That turns out to be true, but that's the second point I want to make.

AUDIENCE: It might be perpendicular.

PROFESSOR: She says there might be one perpendicular to it. And that's the one I was searching for, you're right too. There's going to be a principal axis that's perpendicular to that plane of symmetry, and since we want to define our moments of inertia for this to get started with with respect to g, where will it pass through?

AUDIENCE: Through g.

PROFESSOR: G, kind of by definition. So we're going to have it pass through g. If there's a plane of symmetry, then there is a principal axis perpendicular to it. And we'll0 define it, for the purposes of our discussion, we'll just let it pass through g. It doesn't have to, but that's how we're going about this discussion. Let it pass through the center of mass, this point we call g.

All right, so that means that there's a center of mass in this thing. And I'm just guessing roughly where it is. But in fact if I hung this thing up here like this, and let gravity find it's natural hanging angle, and I drew a plumb line down here, just drew a line that the string would take with a plumb bob. Then I went to some other point and did it again and hung a plumb bob on it drew the line-- where they intersect is the center of mass.

And then you know since it's got symmetry this way that's in the middle. So I guesses that that's about where it is. And there then is a principal axis of this object that's perpendicular to it, passing through the plane-- perpendicular to the plane. And that means now there's two more. But this gets a little more difficult, and I have no clue. There are two more, because the principal axes always come in an orthogonal set. So now that I know one, I know that there's two more somewhere around oriented this way, someplace in this plane.

Maybe one like that, maybe one like this, such that if you spun it about one of those axes, it'd be in balance. You can see that gets a little messy, I can't guess where it is. And so there are ways to find it, one of them would be doing an experiment, seeing which axis it spins nicely around. So that's the second rule. But even just with that one plane of symmetry, you get some pretty good insight.

Now what if there are two planes of symmetry? Yeah?

AUDIENCE: You said that the principal axis does not have to pass through the center of mass?

PROFESSOR: No, I'm saying it doesn't have to pass through the center of mass. The question is?

AUDIENCE: How could it not because it becomes stable [INAUDIBLE].

PROFESSOR: So she's saying it'd be unstable if it's not passing through there. Like if I put an axle through this wheel, and I spun it about some point out here, you don't know because I had that shaker in here, that thing is going to shake like crazy. But does it produce unbalanced torques about my rotation point? It produces centrifugal force that you'll feel like crazy, but does it produce a torque that you'd have to resist with some static torque? What do you think? It won't.

So there's a nuance to this unbalanced thing, and I was going to get to it, probably next lecture, but that is the when we say an object is dynamically balanced, we mean that it doesn't have any unbalanced torques. If it is statically balanced, it's rotating about its center of mass. But you can be statically unbalanced and let it go around this axis, but it'll produce no torques. It's still dynamically balanced.

And the angular momentum of an object which is rotating-- And this we know has a principal axis here, I just moved it off to the side. It's rotating about this. It is dynamically balanced, and if you computed about this point now the Ixy, Ixz off-diagonal entries in that moment of inertia matrix, they're all 0. You'd get no unbalanced torques, but you do have an unbalanced centrifugal force as this thing goes around. And we'll talk a bit more about that.

AUDIENCE: Is that sort of analogous to the homework problem we had a few weeks back with the motorcycle wheel and you just had mass on one side and not on the other?

PROFESSOR: Right. So she's asking about the motorcycle wheel problem where we had that little mass that got there. I think in the next lecture I'm going to come back to that problem just so we could tie a bow around this whole thing and understand why it's unbalanced, how you can balance it, and the difference between static unbalance and dynamic unbalance.

But today, we're talking about symmetry rules. Finally, so I was saying, let's talk about something that has two planes of symmetry. This actually has three planes of symmetry, but we'll settle for two. Pick a plane of symmetry for this object. If I pick-- OK, so she picked the one-- slice it this way. What about the second? Like that. Then a third, right? There's even one like that. So this has three planes of symmetry. But if you have two planes of symmetry that intersect, that are orthogonal to one another, what do you think you can say about that line of intersection?

AUDIENCE: It's the principal axis.

PROFESSOR: It sure is. And it's probably right through g. So if you have two planes of symmetry-- Now make them orthogonal. You can make all sorts of symmetry rules, and I'm just picking these three to help you out. This just to help you see principal axes. If you have two orthogonal planes of symmetry their intersection-- and once you know that, then you go back to rule two. And it tells you everything else you need to know. Because you have one plane of symmetry, you know there is a principal axis perpendicular to it.

Well if you have two planes of symmetry, the rule still holds. There's one perpendicular to each one. The intersection, let's say, of this plane of symmetry and this plane of symmetry is a line which goes right through the center of this thing that way. So there's a principal axis this way. But since there is a plane of symmetry here, there must also be a principal axis perpendicular to it. So sure enough, three principal axes for this thing are through the center, perpendicular this way, perpendicular that way.

You instantly know. Two planes of symmetry-- you instantly know where the three orthogonal principal axes are that past through the center of mass. Yeah?

[? AUDIENCE: Does this all apply ?] just like a constant mass throughout?

PROFESSOR: Not constant, symmetrically distributed density. Right so I'm choosing my words carefully so that I succeed in the following-- that the planes defining mass symmetry will be the same as the planes defining geometric similarity. But you actually don't have to have a constant density, it just has to be distributed so that what I just said is true. So that the geometric symmetries are the same as the mass distribution symmetries.

All right so those are my three rules of symmetry. You could make up others. Those are the three that I've made up to help you see objects. That object, it's a circular disk put on top of another object such that their centers of mass line up. Where are the principal axes of this object using those rules? If you think you know one, tell me.

AUDIENCE: Through the middle.

PROFESSOR: Through the middle of both of them. Probably, good guess. How about another one? Where does this thing have planes of symmetry?

AUDIENCE: So there's a plane of symmetry if you cut it in half. [INAUDIBLE] cut it in half, [INAUDIBLE].

PROFESSOR: OK, and?

AUDIENCE: The other way.

PROFESSOR: One like that, we've got all three. And if we're going to want it to go through the center of mass, then we're going to have to find where the center of mass is this way, but it's about there. So just using the symmetry ideas, you can right away figure out where these principal axes be. And that means from a dynamic point of view, if you spin it about one of those axes, it's nice and dynamically balanced. If you spin it off in some other weird direction is it necessarily dynamically balanced about that axis of spin?

So let me restate that question. We know that this thing has an axis of symmetry principal axis through the center, and another one this way, and another one this way. And if I spin it about any one of those, it's dynamically balanced. But if I pick some other strange direction for the spin, and I spin it about that axis, will I feel unbalanced torques on this axle, on the bearings having to hold this thing in place? Yeah, you better believe it, this thing wobbles like crazy.

So the principal axes are a property of the object, they're not a property of the angular momentum. The angular momentum comes then from multiplying the mass moment of inertia that you've determined times the actual rotation vector. And you'll find out then you get angular components of angular momentum that are not in the direction of spin, and as soon as that happens, you have unbalanced terms.

I've got to get on to something else to help you do homework. So for my disk, with z coming out of the board, the Izz-- so let's say here's x, y, z coming out of the board-- Izz, the mass moment of inertia about this z axis is, from the basic definition, the summation of the mis, xi squared plus yi squared. It's just that for every little mass particle it's the radius squared away from the center of rotation.

That's what the x squared plus y squared is. And we can turn this into an integral. It's the integral of r squared, that distance, times the little mass bit that's there. And that's the same. If you wanted to do the integral as x squared plus y squared dm. But to do this integral for a nice circular, symmetric disk, you can pick a little mass bit that has thickness dr and width rd theta. And this angle here is d theta.

And that's a little bit of area. That's a little dA which has area r dr d theta. It's just length times width. When it's small enough, it's a little rectangle, and it has that area. And it has a volume, dV-- the volume of that thing is just the area times the thickness of it. So here's our disk here, but it has some thickness, and I'll call that h.

So the volume is just h r dr d theta. And the mass, dm, is a density times dV. So I want to integrate this, all I have to integrate the integral then of r squared dm is the integral from 0 to 2pi, 0 to r of rho dV. Rho h-- oh, and I need an r squared-- r squared dV is rho h r dr d theta. So this is 1802 integrals, right?

So is any of this a function of theta? No, so it's a trivial integral, you integrate that over theta, you just get theta, evaluate it 0 to 2pi. So this is 2pi rho h, can all come to the outside, integral 0 to R of r cubed dr. And that ends up-- the r cubed goes to r to the 4th over 4. And the final result of this one is 2pi r to the 4th over 4 rho h, and when new account for h times pi r squared is the volume times rho is the mass.

This all works out to be m r squared over 2. So Izz-- so I needed to do this once for you. For simple things integrate, Izz in this case, you just integrate it out, account for all little mass bits, that is the mass moment of inertia with respect to the axis passing through the center like this. Pardon?

AUDIENCE: [INAUDIBLE]

PROFESSOR: It's this, this is what I'm talking about. Moving on to the last bit. So we need to know how to be able rotate things about places other than their centers of mass. So this is a stick, I can rotate about the center of mass, but it's more interesting if I rotate it about some other point. It makes it a pendulum when I do it around here. So I need to be able to calculate mass moments of inertia about a point that's not through the center of mass.

I know you've seen this before in [? 8.01, ?] so this is going to be a quick reminder. But I'll show you where it comes from. So here's my stick. And it has a center of mass, and that's where G is located here. It has a total length l. I'm going to give it a thickness b, a width a. So it's a stick. a wide, b thick, l long. Uniform has a center gravity right in the middle. And I'm going to attach to this stick-- and this point is kind of hard to draw.

This point is at the center of the stick, OK? I'm going to put my coordinate system attached at the center of gravity, center of mass, and I'm going to make it the-- that's x prime downward, z prime, and y prime is then going off that way. So this is a body set of coordinates at the center of mass. x prime, y prime, z prime. And x happens to be down. And I want to calculate my mass moment of inertia with respect to a point up here that is d, this distance. I've moved up the x-axis an amount d.

I'm going to set a new coordinate system up here. So if this was z prime, my new z is here. It's getting a little messy. Maybe I'll do just a face view. If my previously y prime and x prime were like that, z coming out of the board, now I have a new system that is y and x like this, z still coming out of the board.

Now the coordinate. So how do I calculate mass moment of inertia? Well I want Izz. I probably know Iz'z'. Iz'z' is the mass moment of inertia about this point. I know it's a principal axis from all the things we just-- that square block is the same as this. That's a principal axis in the z prime direction. I know the Izz' with respect to G, I want to know what with respect to this point.

So well Izz, which is my new location up here, and we'll call it A. So Izz here with respect to A is the integral of r squared dm. We've got to do the same integral now. But that's the integral of x squared plus y squared dm. Now I can look at this and I can say oh, well, these are d-- this is separated by d.

I only moved it in the x. The ys haven't moved and the zs didn't change. I just moved my point only in the x direction. So I can now say that in terms of my new coordinate, it's the same as x prime plus d, the distance from here to a point down her, some arbitrary mass point xi is going to be xi' plus d.

So to do this integral in the new coordinates, this is going to be the integral of x prime plus d squared plus y. Now y prime equals y and z prime equals z. Those haven't changed. I didn't move my new coordinate system in the y direction or the z direction. So the coordinate in the new system in y is the same as before. So this is just y prime squared. And this whole thing times d, integrated times every little mass bit.

If I square this, I get x prime squared, 2x'd, d squared, plus y squared. So this integral, Izz with respect to A, when you rearrange it, looks like x prime squared plus y prime squared dm and the integral of a sum is the sum of the integrals. So I break it into bits here, there's a d squared, which is a constant, dm. And then the last term is plus 2d and it's x prime dm.

Just multiply this out, rewrite it, break it apart. Well let's do this one. Integral of x prime squared plus y prime squared dm. That's something that we already have a name for. This is IGzz. It's the original mass moment of inertia with respect to the original coordinate system at G in the z direction. So it's [? IzzG, ?] we already know that. That's given for the object. Plus, this is the integral of dm over the whole extent of the object? Just the mass of the object.

This integral, this is the integral in terms of the x-coordinate. And every mass bit from here, if I go out here and find one, there's an equal and opposite one up here. This is the definition of the center of mass. This integral, if I'm at the center of mass integrating out from it, this is zero because of the definition the center of mass. And I've just proven the parallel axis theorem. Izz about this new point is I about G plus Md Squared, where d squared is the distance I've moved this z-axis to a new place parallel to it.

So Izz with respect to G, the original mass moment of inertia for Izz is m L squared plus a squared over 12. And Ixx m L squared plus b squared over 12. And Izz-- oh, we already know that one. Wait a minute. I haven't told you what that is. That's Izz, Ixx, Iyy.

Just a little messy here. Iyy for this problem is-- I have made a mistake. Ixx is a squared plus b squared. Iyy is m L squared plus b squared. So those are the three-- all with respect to G-- for this stick stick. And I'm going to--

AUDIENCE: [INAUDIBLE]? Is that also divided by 12?

PROFESSOR: Yeah. That kind of sets us up where I can pick up next time. So let's finish by asking ourselves the question, what do we think about-- if I've moved to this new put new position, and I'm not rotating about the center, is this new axis-- this one around the center before, that one we know is a principal axis.

If I rotate about this new place which I've defined the mass moment of inertia about that place? Is this a principal axis?

AUDIENCE: Yes.

AUDIENCE: Yes.

PROFESSOR: How many think yes? How many think no? OK, so lots of people not sure. So my dynamic definition of principal axis is if you can rotate the object about that axis and produce no unbalanced torques, it's a principal axis. And I can do that and this thing will just spin all day long. Now there is a force, you could think of a fictitious force. There's a center of mass out here.

As it spins around, there's a centripetal acceleration making it go in the circle, that means that fictitious force is like there is a centrifugal force pulling out on it. Do I feel that? Do I have to this resist that force as it goes round and round? Yes. So that is an unbalance of a kind we know as a static imbalance, but it doesn't produce torques about my axis right lined up on the center.

AUDIENCE: Doesn't gravity pull on the center of mass [INAUDIBLE].

PROFESSOR: Sure, gravity does, but that's now a different problem. That's what makes this thing act like an oscillator. The torques of the kind I'm talking about is if I compute the angular momentum of this thing and compute dh/dt-- the time rate of change of the angular momentum is a torque on the system, right? I will get the term that makes it spin faster, and I will get, if they exist, terms that make it want to bend this way or bend back.

It only happens if-- if I hold this thing over here, and spin it, get it spinning, and I compute the angle momentum with respect to this point, will I get torques? Yeah, but that's not how I-- that's a different problem. The IG is as if I were computing the angular momentum. Remember I started defining mass moment of inertia matrix based on an angular momentum computation at G. So it's right there in the center of this object, there's no moment arm that is causing torques that's trying to twist this thing about that point.

So the answer to the question is this is a principal axis. Yeah?

AUDIENCE: So if you take the derivative of the angular momentum would you get torques that are not in that direction?

PROFESSOR: If you get torques that aren't in that direction, either you made a mistake doing the math, or you were in error in identifying the mass moment of inertia matrix to begin with. Because if you get torques, there must be non-zero off-diagonal terms and in mass moment of inertia matrix. They are what account for the torques.

So by this parallel axis theorem, any other axis you go to-- if you started at a principal axis, any other axis you create is also a principal axis. That's the movement of just one axis. If you do two, if you move this way and this way, all bets are off. You get a difference answer. And if you're interested in that more complicated problem, read that Williams thing because he does the complete parallel axis, parallel planes and comes up with a super compact little way of calculating them. See you on Thursday.