Chapter 1

Natural Response

A significant portion of these notes are concerned with the study of *finite-dimensional, linear time-invariant* (LTI) systems. We will define this term with more care in section 1.3.2. Such systems can be described by finite-order linear constant coefficient differential equations. Such models are widely applicable to physical systems. In this chapter, we will be primarily concerned with the *natural response* of such models, which is defined as the response which occurs solely from initial conditions with no other inputs. The natural response is also known as the *unforced response* or *characteristic response*. The model differential equation for such a system is homogeneous, in that there is no forcing term.

There is a beautiful property of LTI systems: the homogeneous or natural response can be very simply found. It is composed of weighted sums of functions e^{st}, where s is possibly complex (or most generally such functions multiplied by polynomials in the time variable t). This is a statement about the solution of differential equations. However, it is a remarkable empirical result that such differential equations well-describe many physical systems. Said another way, the types of natural responses discussed below can be easily observed in an experimental context, and in observations of many physical phenomena. The natural response ties things together.

A further surprising result is that real-world systems are frequently able to be represented in terms of very simple models of first- or second-order. When higher-order models are required, these systems have responses composed of sums of first- and second-order responses. So it's very worthwhile to understand the building-block first- and second-order responses in depth.

This chapter is organized as follows: We present first-order systems, and their natural response, starting with a mechanical example. The characteristics of such first-order responses in time are discussed in detail. These responses involve only real functions and thus use only real mathematics. Next we present the similar first-order responses encountered in electrical, thermal, and fluidic systems.

Second-order systems in general have complex-valued natural responses. Thus the section on second-order systems starts with a review of complex numbers. The natural responses for a second-order mechanical system are presented, with individual attention to the overdamped, critically-damped, and underdamped cases. Section 1.2 presents second-order system natural responses. Analogous electrical, thermal, and fluidic second-order systems are discussed next.

Finally, the chapter concludes with a discussion of the natural response of higher-order systems, and a discussion of linearity.

1.1 First-order systems

The canonical[1] homogeneous first-order differential equation is

$$\tau \frac{dy(t)}{dt} + y(t) = 0, \tag{1.1}$$

where we assume $\tau \neq 0$. The variable τ is the system *time constant* and has units of seconds. Here we have explicitly shown the time dependence of $y(t)$. It is also acceptable and more compact to use the form

$$\tau \frac{dy}{dt} + y = 0. \tag{1.2}$$

The response of a such an unforced first-order system is *always* of the form $y(t) = ce^{st}$. This is a simple and beautiful result, easy to remember, and extends to higher-order systems in a natural way. The variable s has units of frequency (sec^{-1}). The differential equation (1.1) will only allow one value of $s = \lambda_1$. We call λ_1 the *characteristic frequency* or equivalently the *eigenvalue* of the system (1.1). In this first-order case, λ_1 is a real number, but in higher-order cases the eigenvalues are more generally complex-valued. The constant c is a real number with the same units as y; it is used to set the value of the function at some point in time, typically $t = 0$. The value at $t = 0$ is called the *initial condition* of the homogeneous response.

You can find the homogeneous solution as follows: First, substitute the assumed form $y(t) = ce^{st}$ into the differential equation. The deriviative

[1] prototypical

operation just brings down a multiplicative term s, and so you have

$$\tau s c e^{st} + c e^{st} = 0. \tag{1.3}$$

This can be factored as

$$(\tau s + 1) c e^{st} = 0. \tag{1.4}$$

Setting the initial condition $c = 0$ satisfies this equation but is not very interesting, since this gives $y = 0$ for all time. The function e^{st} is nonzero for finite s and t, and thus can be divided out to give the *characteristic equation* $(\tau s + 1) = 0$. This has the solution $s = \lambda_1 = -1/\tau$, which is the one and only characteristic frequency (eigenvalue)[2] associated with this first-order system.

Thus we have arrived at the homogeneous solution

$$y(t) = c e^{-\frac{t}{\tau}}. \tag{1.5}$$

The response decays to zero with increasing time if $\tau > 0$; if the natural response of a system always decays to zero with increasing time for any initial conditions, we say that the system is *stable*. If the response goes off to infinity with increasing time for some initial conditions, the system is *unstable*.

The response (1.5) has the initial value $y(0) = c$. The graph of this response is shown in Figure 1.1 for an initial condition $c = 1$ with the four values $\tau = 2, 1, 0.5, 0.1$. As you can see, τ represents the characteristic time for the response to decay toward zero; smaller values of τ correspond with faster responses.

Because such a response is widely applicable in real engineering systems, we will take a bit of time to understand it in more depth. Your efforts here to internalize an understanding of this response and its characteristics will pay dividends throughout your engineering studies and practice. Specifically, in an interval of one time constant, the response shown decays to a value of 0.37 times the value at the start of the interval. This is so because $e^{-1} = 0.3679 \approx 0.37$. Since this response has an initial value of 1, the response decays to a value of 0.37 in one time constant, 0.37^2 in two time constants, and a value of 0.37^n in n time constants. You should verify this result to graphical accuracy for all four of the time constant values; they pass through the dashed line $y = 0.37$ in an interval equal to τ. And, in 3 seconds, the response for $\tau = 1$ sec passes through a value of $0.37^3 \approx 0.05$; we would

[2]The terms *characteristic frequency* and *eigenvalue* are equivalent, and will be used interchangeably herein.

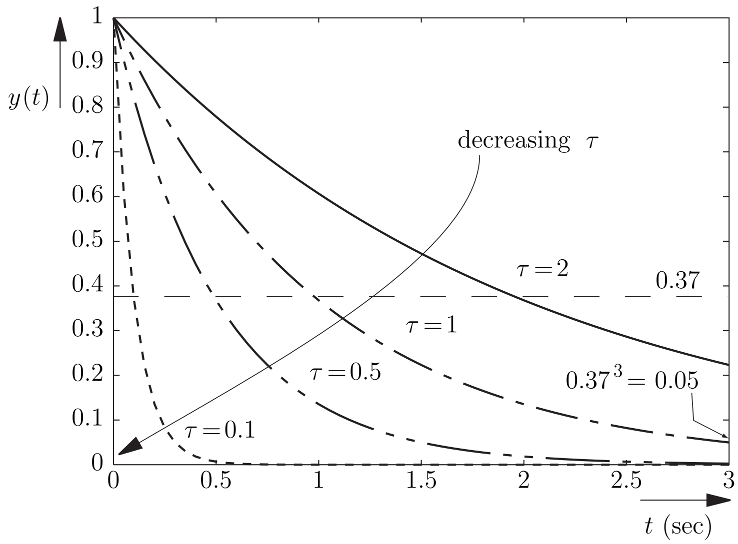

Figure 1.1: First-order system response to an initial condition $c = 1$ with the four values $\tau = 2, 1, 0.5, 0.1$.

say that this response has settled to within 5% in 3 time constants. How many time constants would it take to settle to within 1%? (You should be sure you can answer this before going on.) Meanwhile, in 3 seconds, the $\tau = 0.1$ sec response has passed through 30 time constants, and has a value of $e^{-30} = 9.4 \times 10^{-14}$. This is pretty close to zero, but in theory the response never quite gets to zero, no matter how long you wait; it just keeps decaying by further factors of 0.37.

So we see that the eigenvalue captures the time-scale of the first-order response. This idea extends to the higher-order systems considered later. In these more general cases, the eigenvalues may have imaginary components; in the first-order case considered above they are pure real. Because of the primary importance of the eigenvalues of a system, it is common in practice to graphically plot the eigenvalue locations on a plane with horizontal axis $\text{Re}\{s\}$, and vertical axis $\text{Im}\{s\}$. This complex plane is referred to as the s-plane, and the eigenvalue locations are called *poles*. For example, in the first-order system considered above, the eigenvalue is $\lambda_1 = -1/\tau$; this system is thus also said to have a pole at $s = \lambda_1 = -1/\tau$. In the complex plane, poles are plotted as x's; for the first-order system, the pole diagram

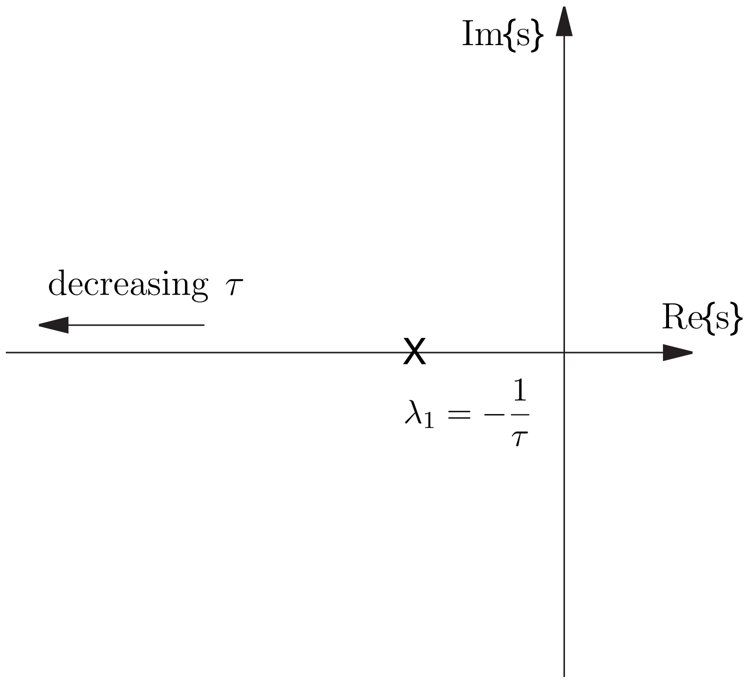

Figure 1.2: First-order system pole location as a function of τ. Arrow indicates the direction of decreasing τ.

appears as shown in Figure 1.2. Decreasing values of τ result in the pole moving to the left along the negative real axis. Thus, faster systems have poles located further from the origin in the s-plane.

A differential equation of the form (1.1) occurs as the mathematical model for systems of many different physical principles. In the sections below, we show the process of modeling first-order systems from the mechanical, electrical, thermal, and fluidic domains. In these domains, homogeneous responses such as shown in Figure 1.1 occur with a variety of associated physical units. The beauty of system theory is first that it is found to be applicable to many classes of real-world systems, and secondly that we can thereby understand these systems from a common mathematical framework.

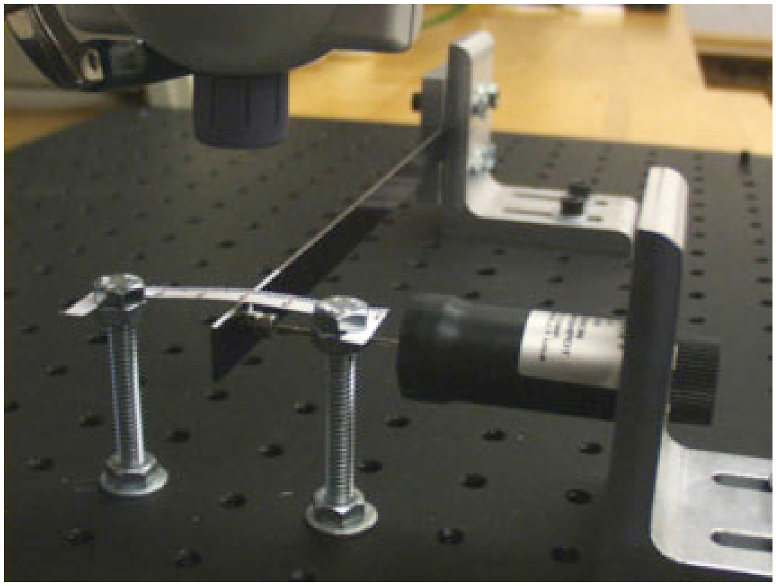

Figure 1.3: Picture of mechanical system which can be modeled as first-order.

1.1.1 Mechanical translational first-order system

Consider the mechanical system shown in the picture of Figure 1.3 as used in Lab 1. This consists of a spring-steel beam rigidly fixed at one end, and attached to an air cylinder damper on the other end. We will consider this as a translational system, with the point of translation corresponding to the nearly straight line motion of the end of the beam where it joins the air piston damper. The air piston damper[3] consists of a graphite piston sliding in a precisely fit glass cylinder as shown in Figure 1.4. The knob at the near end controls an adjustable orifice to set the resistance to flow in and out of the damper, and thereby set the damping coefficient.

Figure 1.5 shows experimental data taken from this system via videotaping at 20 frames per second, as well as data from a model adjusted to match this response. The measured data points are shown in blue, with asterisks at the data points taken every 1/20th of a second. The red curve is a plot of the first-order response (1.5) with the parameters adjusted to reasonably

[3]Also known as an Airpot, which is a trademark of the Airpot Corporation, Norwalk, CT.

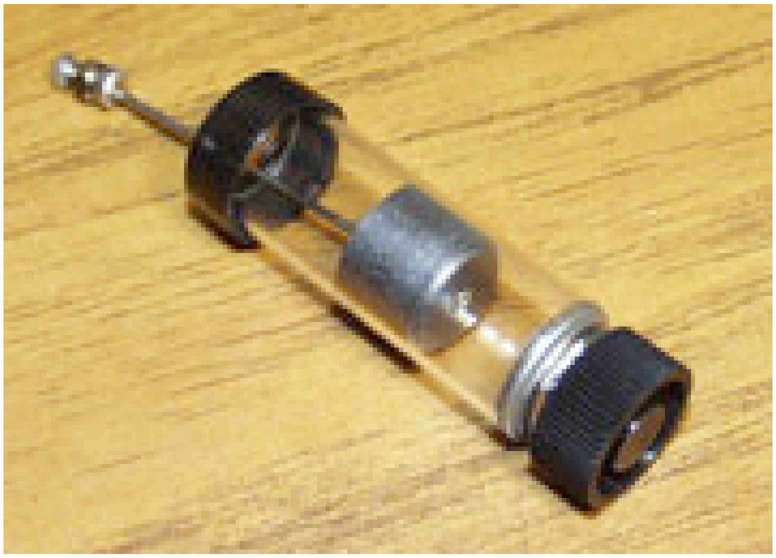

Figure 1.4: Picture of air piston damper.

fit the data. The fitted model response is

$$y(t) = 1.5 \times 10^{-2} e^{-1.65t} \quad [\mathrm{m}], \tag{1.6}$$

and thus the time constant is $\tau = 1/1.65 = 0.61$ sec. The initial condition is $c = 1.5$ cm.

The first-order model (1.5) fits this response very well. The experimental data is a bit noisy as might be expected. The primary noise source is that the video camera frame rate is not very constant. This could be improved with better video hardware, but is not important for this experiment.

The simplest lumped mechanical model which fits this response is the first-order mechanical spring-damper system shown in Figure 1.6. Here we assume that the link can only move in the x-direction. The cantilever beam acts as a spring which is linear for moderate deflections. The spring constant k for this beam can be calculated from first principles. With this calculated spring constant we can compute the damping coefficient equivalent b for the air piston damper.

As shown in the figure, the system consists of a spring and damper attached to a rigid massless link. The link represents the connection between the spring and damper, but contributes no dynamics of its own. The position of the link is denoted as x. The zero of position is indicated in the figure by the vertical line connecting to the arrow which indicates the direction

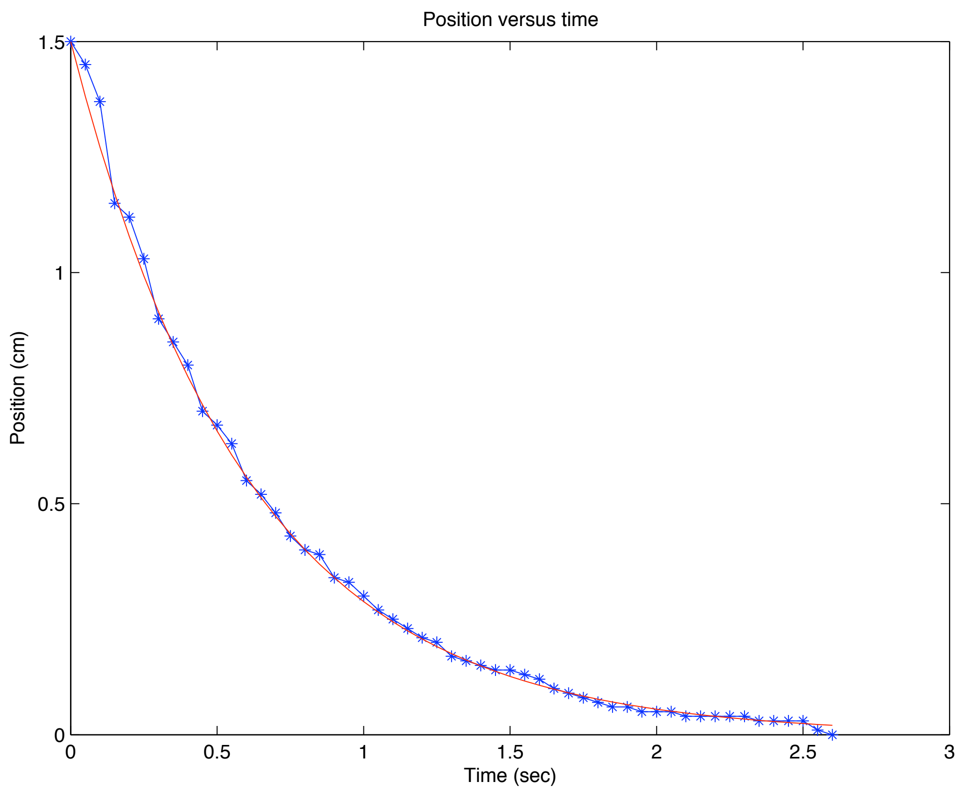

Figure 1.5: Experimental natural response of beam/air piston system, and first-order model response.

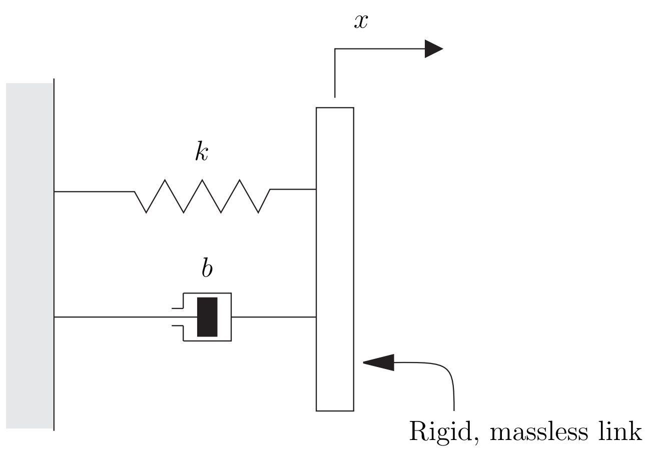

Figure 1.6: First-order mechanical system model.

of increasing x. This choice of zero accounts for the rest position (zero-force length) of the spring. The spring is moved by a force proportional to motion in the x-direction, $F_k = kx$. The damper is moved by a force which is proportional to velocity in the x-direction, $F_b = b\,dx/dt$.

Newton's second law states that $F = ma = m\ddot{x}$, where F is the sum of the forces acting on a mass. This relationship also applies to the massless link, but since the link is massless, the forces must instantaneously sum to zero. For any mass element, or massless assembly from a system, Newton's second law can be captured in the form of a free-body diagram. For this system the free-body diagram appears as shown in Figure 1.7.

Summing forces acting on the link and applying Newton's second law yields the system equation of motion

$$-F_k - F_b = -kx - b\frac{dx}{dt} = 0. \tag{1.7}$$

The minus signs appear here for the forces F_k and F_b since they act on the link in the $-x$-direction. The zero term on the right is due to the fact that the link is massless. The governing differential equation can be rewritten as

$$\frac{b}{k}\frac{dx}{dt} + x = 0, \tag{1.8}$$

If we define $\tau = b/k$, this is in the form of (1.1). The natural response is thus as calculated in section 1.1, with its associated figures.

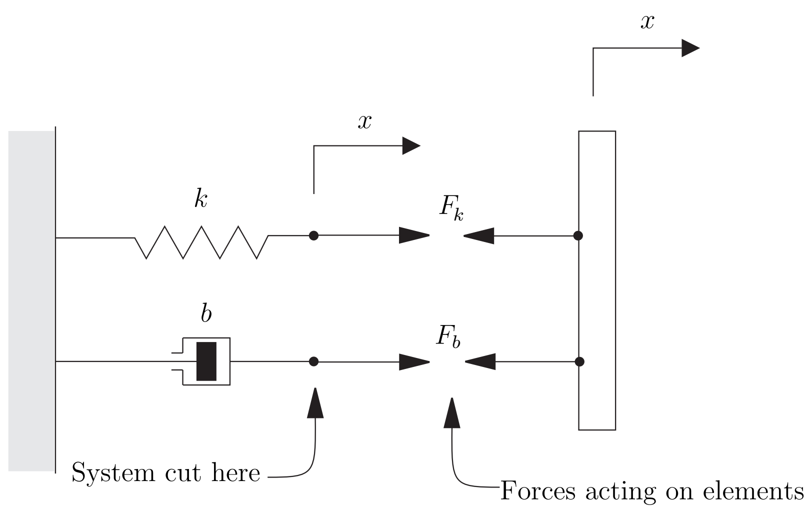

Figure 1.7: Free body diagram for massless link of first-order system.

We can calculate that for the dimensions of this beam $k = 170$ N/m. With this value in hand the model damping coefficient is given by

$$b = k\tau = 170 \cdot 0.61 = 104 \quad \text{[N sec/m]} \tag{1.9}$$

Dynamic systems can be studied at a number of levels of detail. Models of greater complexity could be readily justified for the beam air pot system if it were studied in more depth. For example, the distributed mass and compliance of the beam would lead to the existence of vibratory modes on the beam itself. These modes would require a high dimensional or infinite dimensional model that could more accurately capture some of the transient behavior of the beam. Further, we have ignored the compressibility of the air in the cylinder of the damper. With finite compressibility, the air pot is a thermodynamic system in that the temperature of the air contained within the cylinder is a reflection of the work done on the air by the piston, as well as inputs/losses of heat from the outside world. Such considerations are important to understand the behavior in many cases, but are well outside the scope of topics for this text. You need to study fundamentals of thermodynamics to fully understand this issue.

Meanwhile, within the right range of time scales and accuracy requirements, a simple first order lumped model well-captures the dominant dy-

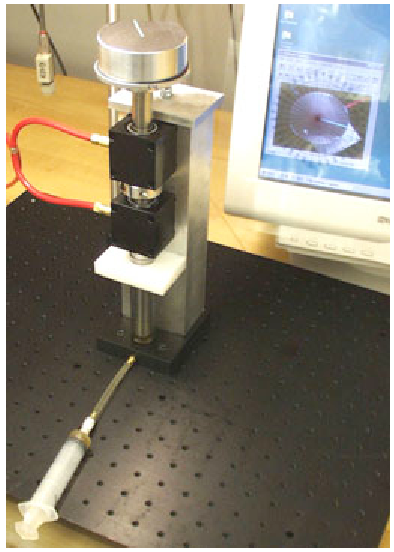

Figure 1.8: Picture of mechanical rotational system which can be modeled as first-order.

namics of the airpot/beam system, as verified by the experimental data shown above. For information on expanding models to include such additional detailed effects, in this and many other systems, take a look at the Master's thesis of Katie Lilienkamp [1]. In particular, the beam/air-piston system is treated in great detail in section 3.3 of this reference.

1.1.2 Mechanical rotational first-order system

Consider the mechanical rotational system shown in the picture of Figure 1.8. This system is described in more detail on the Activlab pages under the heading of Lab 2; you can see a video of it in motion on these pages. This system consists of a shaft rotating about a vertical axis. The axis of rotation is constrained by a pair of *air bearings*, which use pressurized air to create a nearly-frictionless rotational/translational bearing. Since the air bearings do not constrain axial motion, the shaft rests on a ball bearing resting on a hardened flat. This ball-on-flat acts as the axial bearing for the system. If the rotational axis is properly aligned perpendicular to the flat, then this axial bearing exhibits very little friction.

The rotating shaft carries a brass flywheel which serves as an additional rotational inertia. This flywheel can be placed on the hub to increase the inertia, or removed to decrease the inertia. Figure 1.9 shows the brass

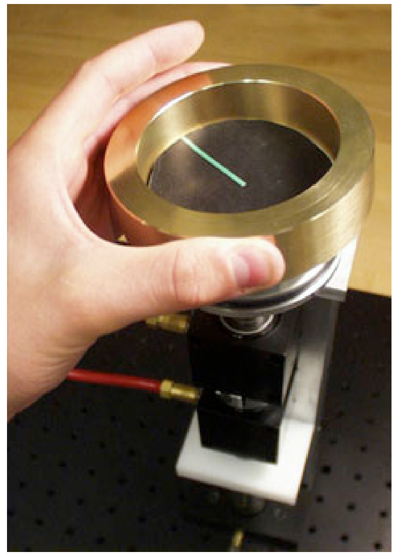

Figure 1.9: Picture of brass flywheel being placed on top of shaft of rotational system.

flywheel being put in place on the top of the hub.

A line drawing of the system is shown in Figure 1.10. Here we can observe the shaft located in the air bearings. The axis of rotation is vertical in this figure. At the top of the shaft is the flywheel hub which is shown with the brass weight removed. At the bottom of the shaft there is a cup filled with a viscous liquid. In the present case this liquid is honey. More detail of the bottom end of the shaft is shown in Figure 1.11. Here you can see the ball bearing which is mounted on to the end of the shaft and rotates with the shaft. The ball bearing rests on the hardened flat shown at the bottom of the figure. Honey is filled within the chamber to a depth L and has an annular thickness t.

Figure 1.12 shows experimental data taken from this system. The response shown in the figure looks reasonably modeled as first-order. At the level of modeling that we require, we can then think of this system as composed of a rotational inertia spinning on a rotational damper a shown in Figure 1.13.

The rotational equivalent of Newton's second law is $\sum \tau = J\dot{\omega}$. The only torque acting on the inertia is due to the the viscous drag of the rotational damper $\tau = -b\omega$. Summing torques acting on the inertia yields the

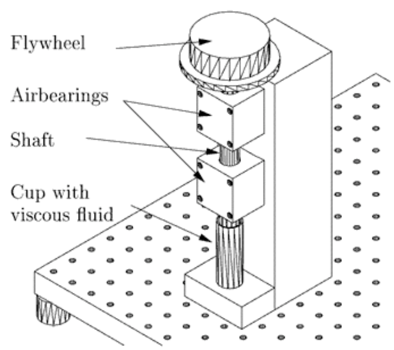

Figure 1.10: Line drawing of rotational system.

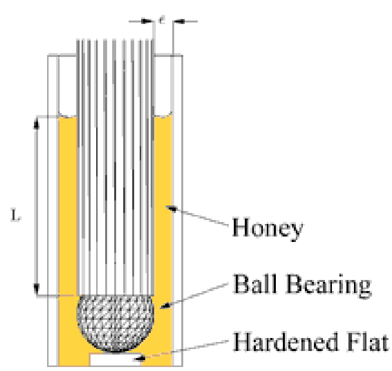

Figure 1.11: Cross-section at bottom of shaft showing ball bearing on flat, and honey used for viscous damping.

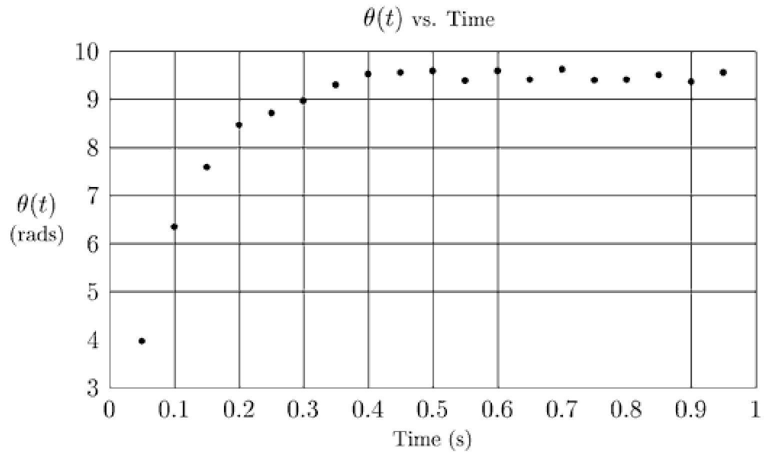

Figure 1.12: Experimental natural response of mechanical rotational system.

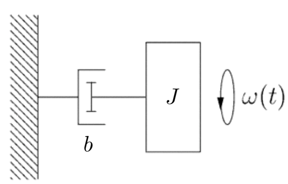

Figure 1.13: Model for rotational system.

differential equation.

$$J\frac{d\omega}{dt} + b\omega = 0 \tag{1.10}$$

which can be rewritten in standard form as

$$\frac{J}{b}\frac{d\omega}{dt} + \omega = 0 \tag{1.11}$$

This equation has the solution

$$\omega(t) = ce^{-t/\tau} \tag{1.12}$$

if we define $\tau = J/b$ [sec]. Thus the system has an eigenvalue $\lambda_1 = -\frac{1}{\tau} = -\frac{b}{J}$. Integrating this result allows us to solve for the associated angular position as

$$\int_0^t \omega dt = \int_0^t ce^{-t/\tau} \tag{1.13}$$

$$\theta(t) - \theta(0) = -\tau ce^{-t/\tau}\Big|_0^t \tag{1.14}$$

$$= -\tau c[e^{-t/\tau} - 1] \tag{1.15}$$

$$= \tau c[1 - e^{-t/\tau}] \tag{1.16}$$

which gives

$$\theta(t) = \theta(0) + \tau c[1 - e^{-t/\tau}]. \tag{1.17}$$

To graphical accuracy, the experimental data of Figure 1.12 is reasonably well fit by the function

$$\theta(t) = 1 + 8.5[1 - e^{-t/0.1}] \text{ rad}, \tag{1.18}$$

that is, with $\tau = 0.1$ sec, $\theta(0) = 1$ rad, and $c = 85$ rad/sec. This allows us to give the estimated velocity as a function of time as

$$\omega(t) = 85e^{-t/0.1} \text{ rad/sec}. \tag{1.19}$$

At this point we could develop a calculation of the rotational system inertia from first principles. If we know the rotational inertia of this system, we can then use the time constant result $\tau = J/b$ to calculate the equivalent rotational damping. Alternately, we could experimentally measure the rotational damping and thereby develop an estimate of the rotational inertia J.

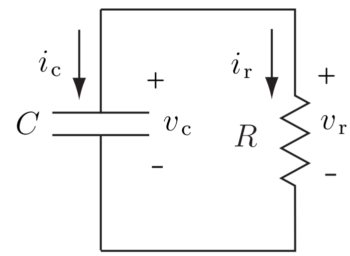

Figure 1.14: First-order parallel RC circuit diagram.

1.1.3 Electrical first-order system

The circuit shown in Figure 1.14 is a parallel RC circuit which can be described by a first order differential equation. The formulation of the differential equation goes as follows. First we need to account for each of the network elements. The resistor has a current-voltage relationship described by Ohm's law $v_r = i_r R$. The capacitor has a current-voltage relationship given by $i_c = C\frac{dv_c}{dt}$.

The currents i_r and i_c must be equal and opposite so that their sum is equal to zero, since current cannot accumulate at their common node. That is, we must have

$$0 = i_r + i_c = \frac{v_r}{R} + C\frac{dv_c}{dt}. \tag{1.20}$$

Recognize further that since the two elements are connected in parallel, their voltages must be equal: $v_r = v_c$. You substitute this into (1.20), and multiply through by R to find

$$RC\frac{dv_c}{dt} + v_c = 0. \tag{1.21}$$

If we define $\tau = RC$, this is in the form of (1.1). The natural response is thus as calculated in section 1.1, with its associated figures. Specifically, if the initial voltage on the capacitor is defined as $v_c(0) = v_0$, then the voltage as a function of time varies as

$$v_c(t) = v_0 e^{-t/RC} \quad \text{[Volts]}. \tag{1.22}$$

For example, suppose that we set $C = 100\ \mu$F and $R = 1$ MΩ. Then the time constant is $\tau = 100$ sec; it will take the capacitor voltage 100 seconds to decay to 37% of its initial value.

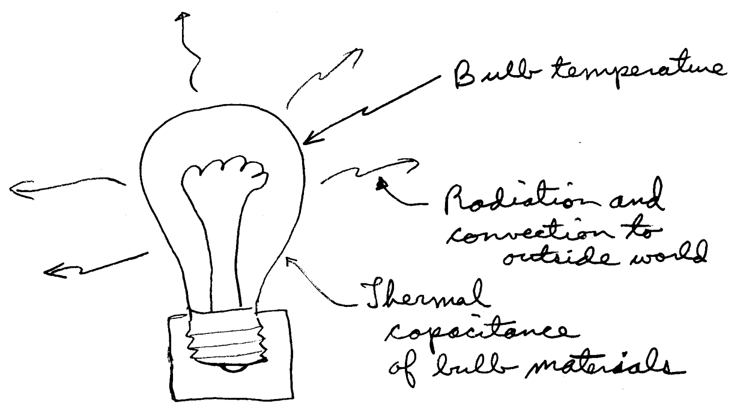

Figure 1.15: Sketch of bulb and relevant thermal elements.

1.1.4 Thermal first-order system

For an example thermal system we study the desk lamp shown in the picture (to be added). This lamp bulb is electrically heated via the bulb filament. The resulting bulb temperature is measured with the infrared sensor shown in the figure (to be added). A sketch of the light bulb in the lamp is shown in the line drawing of Figure 1.15.

We left the lamp on for a long enough time to reach steady state, and then turned off the lamp and measured the decay of temperature back to ambient. Data taken from this system is shown in tabular and graphical form in Figure 1.16 By inspection of this data, the bulb system is well-fit by a first-order model of the form of (1.1). An estimate of the associated time constant is about 3 minutes. But we need to have τ in seconds, so the system time constant is formally given as $\tau = 180$ sec.

An abstraction to a lumped model of this system is shown in Figure 1.17. Here the *thermal capacitance* of the bulb is summarized by the block of material labeled with the capacitance C_b with units of [J/°K]. The block is assumed to have a uniform temperature T_b [°K]. This block has a total stored thermal energy $W_b = C_b T_b$ [J]. The change of thermal stored energy happens via heat flow

$$q_b = \frac{dW_b}{dt} = C_b \frac{dT_b}{dt}. \tag{1.23}$$

Here q_b in units of watts represents heat flow *into* the bulb. As shown in the figure, we assume that the block is insulated on three sides, and so the

Hot light bulb cooling

t (min)	T (deg C)
0	101.2
1	81.6
2	66.2
3	54.4
4	47.6
5	41.8
6	39.4
7	35.8
8	34.4
9	32.6
10	32.2
12	29.4
14	29.4

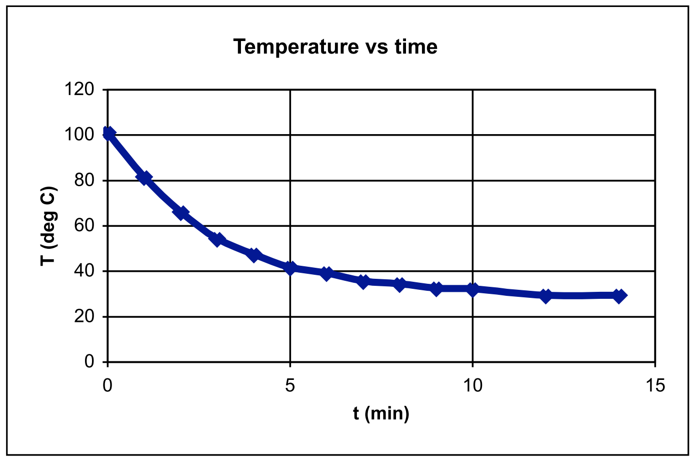

Figure 1.16: Data from light bulb cooling experiment.

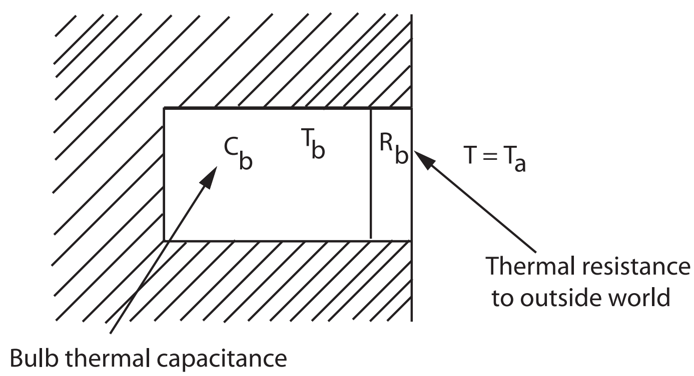

Figure 1.17: Lumped model for bulb cooling experiment.

heat flow through those sides is zero. The block is connected to the outside ambient temperature via the thermal resistance R_b, such that

$$q_b = \frac{T_a - T_b}{R_b}. \tag{1.24}$$

This resistance represents the flow of heat into the bulb as a linear function of the temperature difference[4] between the ambient and the bulb temperatures.

Setting equality between the last two equations gives

$$C_b \frac{dT_b}{dt} = \frac{T_a - T_b}{R_b}. \tag{1.25}$$

Now, it's convenient to define a variable to represent the temperature difference between the bulb and ambient: $T \equiv T_b - T_a$. Since the ambient temperature is constant, $dT/dt = dT_b/dt$. Making these substituations and multiplying (1.25) through by R_b yields

$$R_b C_b \frac{dT}{dt} + T = 0. \tag{1.26}$$

If we define $\tau = R_b C_b$, this is in the form of (1.1). The natural response is thus as calculated in section 1.1, with its associated figures. Specifically, if the initial temperature difference of the bulb is defined as $T(0) = T_0$, then the temperature difference as a function of time varies as

$$T(t) = T_0 e^{-t/R_b C_b} \quad \text{[K]}. \tag{1.27}$$

If you want to convert back to the absolute temperature of the bulb, remember that $T_b = T + T_a$.

1.1.5 Fluidic first-order system

A fluidic system which can be modeled with a first-order differential equation is shown in Figure 1.18. Here a tank filled with liquid drains through a long, thin pipe. The height of the liquid above the pipe inlet is defined as h. If we assume that the liquid has a density of ρ [kg/m^3], then the pressure P_t at

[4] In real systems, more exact and likely nonlinear models can apply, but a linear model gives a first understanding of this system response, and is well able to match the measured behavior. For example, pure radiative cooling varies as temperature difference to the fourth power, which is highly nonlinear. There will certainly be significant radiative heat flow in this system, however, the experimental data fits well to a linear heat flow model which suggests that radiative cooling is not highly significant at the bulb envelope temperatures of 100 $^\circ C$.

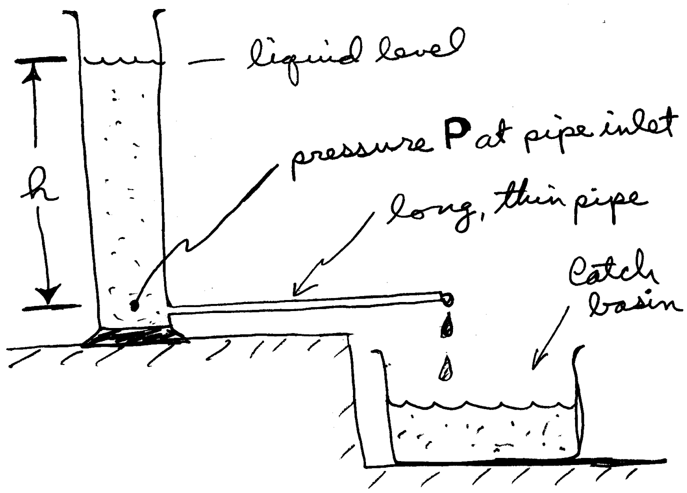

Figure 1.18: Liquid tank experiment.

the inlet of the pipe is given by $P_t = P_a + \rho g h$ [N/m^2]. In the SI system of units, the units of pressure are Pascal's, *i.e.*, 1 Pa = 1 N/m^2. Here P_a is the ambient pressure outside the system, and g is the acceleration of gravity. The pipe volumetric flow into the tank is defined as q_t [m^3/s]. The flow is assumed to vary linearly with the pressure difference as

$$q_t = \frac{P_a - P_t}{R} = \frac{-\rho g h}{R}. \tag{1.28}$$

Here R [$\mathrm{Pa} \cdot \mathrm{s/m^3}$] is the fluidic resistance of the pipe.

If we assume that the tank has a constant cross-sectional area A, then the fluid height varies with flow into the tank as

$$\frac{dh}{dt} = \frac{q_t}{A} \tag{1.29}$$

We multiply through by A and set equality between the last two equations to give

$$\frac{RA}{\rho g}\frac{dh}{dt} + h = 0. \tag{1.30}$$

If we define $\tau = RA/\rho g$, this is in the form of (1.1). The natural response is thus as calculated in section 1.1, with its associated figures. Specifically, if the initial fluid height is defined as $h(0) = h_0$, then the fluid height as a function of time varies as

$$h(t) = h_0 e^{-t\rho g/RA} \quad [\mathrm{m}]. \tag{1.31}$$

1.2 Second-order systems

In the previous sections, all the systems had only one energy storage element, and thus could be modeled by a first-order differential equation. In the case of the mechanical systems, energy was stored in a spring or an inertia. In the case of electrical systems, energy can be stored either in a capacitance or an inductance. In the basic linear models considered here, thermal systems store energy in thermal capacitance, but there is no thermal equivalent of a second means of storing energy. That is, there is no equivalent of a thermal inertia. Fluid systems store energy via pressure in fluid capacitances, and via flow rate in fluid inertia (inductance).

In the following sections, we address models with two energy storage elements. The simple step of adding an additional energy storage element allows much greater variation in the types of responses we will encounter. The largest difference is that systems can now exhibit oscillations in time in their natural response. These types of responses are sufficiently important that we will take time to characterize them in detail. We will first consider a second-order mechanical system in some depth, and use this to introduce key ideas associated with second-order responses. We then consider second-order electrical, thermal, and fluid systems.

1.2.1 Complex numbers

In our consideration of second-order systems, the natural frequencies are in general complex-valued. We only need a limited set of complex mathematics, but you will need to have good facility with complex number manipulations and identities. For a review of complex numbers, take a look at the handout on the course web page.

1.2.2 Mechanical second-order system

The second-order system which we will study in this section is shown in Figure 1.19. As shown in the figure, the system consists of a spring and damper attached to a mass which moves laterally on a frictionless surface. The lateral position of the mass is denoted as x. As before, the zero of