18.S997 Notes

The Elekes-Sharir Approach to the Distinct Distance Problem

Today's the last background lecture in incidence geometry. We'll discuss one of the latest methods to approaching the distinct distance problem, which has a cool connection to incidence geometry.

Before going into that, let's review how we were thinking about the distinct distance problem. Suppose we have N points and $t \ll N$ distances. Draw tN circles around each point, and consider the circles as arcs. We tried this approach before, but never used the fact that the radii at each of the points must be the same.

We make some new definitions to take advantage of that. Let $P \subset \mathbb{R}^2$ be the set of points and $d(P)$ the set of nonzero distances. Let $Q(P) = \{(p_1, q_1, p_2, q_2) \in P^4 : |p_1 - q_1| = |p_2 - q_2| \neq 0\}$. We'd expect $|Q(P)|$ to be large.

Lemma 1. $|d(P)|\,|Q(P)| \geq (N^2 - N)^2 \gtrsim N^4$.

Proof. Let $d(P) = \{d_1, \dots, d_s\}$ where $s = |d(P)|$. We can just count $n_j = \{(p,q) \in p^2 : |p-q| = d_j\}$ and $\sum_j n_j = N^2 - N$. If we pick the distance first and then choose two pairs equalling that distance, we get $|Q(P)| = \sum_j n_j^2$. Then we just use Cauchy-Schwarz: $N^2 - N = \sum_{j=1}^{s} n_j \cdot 1 \leq \left(\sum n_j^2\right)^{1/2} s^{1/2} = |Q(P)|^{1/2}\,|d(P)|^{1/2}$, as desired. □

This is not at all surprising: If there are few distances, then there should be a lot of quadruples. So we'd also like to count these quadruples in another way, and figuring out a way to do this was their key insight.

Let G be the group of orientation-preserving rigid motions of the plane.

Lemma 2. $|p_1 - q_1| = |p_2 - q_2| \neq 0$ *iff* $\exists! g \in G$ *with* $g(p_1) = p_2$ *and* $g(q_1) = q_2$.

That got people thinking about which rigid motions take p_1 to p_2. Let $S_{p_1,p_2} = \{g \in G : g(p_1) = p_2\}$. This is a 1-dimensional curve in G, which is a 3-dimensional Lie group.

Lemma 3. *Assume* $p_1 \neq q_1$. *Then* $|p_1 - q_1| = |p_2 - q_2|$ *iff* $|S_{p_1,p_2} \cap S_{q_1,q_2}| = 1$, *and* $|p_1 - q_1| \neq |p_2 - q_2|$ *iff* $|S_{p_1,p_2} \cap S_{q_1,q_2}| = 0$.

So we can look at the incidence geometry of these curves in G. If a point lies on two curves, it corresponds to two quadruples, and if a point lies on three curves, it corresponds to six quadruples. Let $G_{=k} := \{g \in G :$ g lies in exactly k curves of $S\}$.

Let $E : Q(P) \to g$ be given by Lemma 3. Then the image of E is contained in $\bigcup_{k \geq 2} G_{=k}$. We have $E^{-1}(g) = 2\binom{k}{2}$, since we can take the quadruples in either order. Therefore, we have

Lemma 4. $|Q(P)| = \sum_{k=2}^{N} |G_{=k}|\, 2\binom{k}{2}$.

We usually calculated things with $G_k = \{g \in G : g$ lies in $\geq k$ curves$\}$. So writing in terms of these, we have

Lemma 5. $Q(P) \sim \sum_{k=2}^{N} |G_k|\, k$.

Proof. We have

$$
\begin{aligned}
|Q(P)| &= |G_{=k}|\, 2\binom{k}{2} \\
&= \sum_{k=2}^{N} [|G_k| - |G_{k+1}|](k^2 - k) \\
&= \sum_{l} |G_l| \left[(l^2 - l) - ((l-1)^2 - (l-1))\right] \\
&\sim \sum_{l} l\, |G_l| \,.
\end{aligned}
$$

□

We also have this other characterization of G_k:

Lemma 6. $G_k = \{g \in G : |gP \cap P| \geq k\}$.

This is sort of a generalization of symmetries, where we'd require $gP = P$. So we can think of these as partial symmetries.

Example. Suppose our set of points is an $s \times s$ square grid with $N = s^2$. Then $|G_{s^2}| = 4$. What about things like $\left|G_{\frac{1}{10}s^2}\right|$? Well, it takes a while to explain, but $|G_k| \sim N^3 k^{-2}$ for all $2 \leq k \leq \frac{1}{10}N$.

That this is the best you can do was a conjecture:

Conjecture (ES1)**.** *If $P \subset \mathbb{R}^2$ with $|P| = N$ and $2 \leq k \leq n$, then $|G_k| \lesssim N^3 k^{-2}$.*

This has since been proven, and we'll prove it in this class using the polynomial method.

Let's see the consequences. $|Q(P)| \leq \;{}^{N}_{k=2} |G_k|\, k \lesssim \;{}^{N}_{k=2} N^3 k^{-1} \lesssim N^3 \log N$. Then $|d(P)| \gtrsim N^3 / |Q(P)| \gtrsim N/\log N$. We'll prove this whole chain of implications using the polynomial method. We see that we've claimed that the conjecture itself is sharp for the square grid, so we know that the square grid indeed does have that many quadruples. But we could have lost some at the last step because we used Cauchy-Schwarz, and indeed, we checked earlier that there are $N/\sqrt{\log N}$ distinct distances in the large square grid.

Let's get a better feel for these rigid motions. We first have the translations T, which are congruent to the plane $\mathbb{R}^2$.

Lemma 7. $|T \cap G_k| \lesssim N^3 k^{-2}$.

Proof. Consider the number of translation quadruples $Q_T \subset Q(P) = \{(p_1, q_1, p_2, q_2) \in P^4$ such that $p_1 - q_1 = p_2 - q_2 = 0\}$. Then $\# Q_T \leq N^3$ because $\forall p_1, q_1, p_2$, there is at most one choice of p_2. Then define $E : Q_T \to T$ similar to before, and if $g \in G_k$, $\left|E^{-1}(g)\right| \sim k^2$. So $|Q_T| \geq |G_k \cap T| \cdot 2 \; {k \atop 2}$, and we're done. □

Now we'd like to "straighten" $G' := G/T$. There's a way to do this to make it correspond to the incidence geometry of points and lines. G' is a rotation around a fixed point $(x, y) \in \mathbb{R}^2$ by angle $\theta \in (0, 2\pi)$. Then we define $\rho : G \to \mathbb{R}^3$ by $\rho(x, y, \theta) = (x, y, \cot \theta/2)$.

Proposition. *$\rho(S_{pq} \cap G')$ is a line ℓ_{pq}.*

Proof. Indeed, if we rotate from p to q, the point of rotation must be on the perpendicular bisector of p and q. It's just trigonometry from here. □

In fact, we have the following:

Proposition. *Let $v = \;\frac{p_2 - q_2}{2}, \frac{q_1 - p_1}{2}\;$ be a vector perpendicular to $p - q$ with length $\frac{1}{2}|p - q|$ and $a = \frac{p+q}{2}$. Then $\rho(S_{pq} \cap G')$ is a line parameterized by $\ell_{pq} : t \mapsto (a + tv, t)$.*

Let $\mathcal{L} = \{\ell_{pq}\}_{p,q \in P}$, N^2 lines. Then $|G'_k|$ is the number of points in $\geq k$ lines of $\mathcal{L}$. Remember that the incidence geometry depended on whether they were all in the plane.

Lemma 8. *If $q = r$ then ℓ_{pq} and ℓ_{qr} are skew.*

Proof. $S_{pq} = \{g \in G : g(p) = q\}$, so $S_{pq} \cap S_{pr} = \emptyset$. This shows that $\ell_{pq} \cap \ell_{pr} = \emptyset$, so we also have to show they aren't parallel. The "slope" $((dx/dz, dy/dz))$ of ℓ_{pq} is $v(p,q)$ and these slopes are different. $\square$

We also realized that there was a problem if too many lines lay in some regulus. We won't prove this in class today but defer this proof to a while later, but there are $\leq N$ lines of $\mathcal{L}$ in any degree 2 surface.

Conjecture (ES2A)**.** *If $\mathcal{L}$ is a set of L lines with at most $L^{1/2}$ in any plane or degree 2 surface, then $|P_2| \lesssim L^{3/2}$.*

Conjecture (ES2B)**.** *If $\mathcal{L}$ is a set of L lines with at most $L^{1/2}$ in any plane and $3 \leq k \leq L^{1/2}$, then $|P_k| \lesssim L^{3/2} k^{-2}$.*

Finally, we saw another log-log graph of the bounds we had. We had the S-T bound for any L lines that was piecewise linear with two regions, from 2 to $L^{1/2}$ and $L^{1/2}$ to L. Then if we assume the number of lines in any plane or degree 2 surface is small, then we lower the first line.

This finishes our background on incidence geometry. In the next session, we'll pick up with the polynomial method.