MY PUTNAM PROBLEMS

These are the problems I proposed when I was on the Putnam Problem Committee for the 1984–86 Putnam Exams. Problems intended to be A1 or B1 (and therefore relatively easy) are marked accordingly. The problems marked with asterisks actually appeared on the Putnam Exam (possibly reworded). — R. Stanley

1. (A1 or B1 problem) Given that

$$\int_0^1 \frac{\log(1+x)}{x} dx = \frac{\pi^2}{12},$$

evaluate

$$\int_0^1 \int_0^y \frac{\log(1+x)}{x} dx\, dy.$$

2.* (A1 or B1 problem) Let B be an $a \times b \times c$ brick. Let C_1 be the set of all points p in $\mathbb{R}^3$ such that the distance from p to C (i.e., the minimum distance between p and a point of C) is at most one. Find the volume of C_1.

3.* (A1 or B1 problem) If n is a positive integer, then define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(n)$ and $Q(n)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

for all $n \geq 1$.

4. (A1 or B1 problem) Let C be a circle of radius 1, and let D be a diameter of C. Let P be the set of all points inside or on C such that p is closer to D than it is to the circumference of C. Find a rational number r such that the area of P is r.

5. Let n be a positive integer, let $0 \leq j < n$, and let $f_n(j)$ be the number of subsets S of the set $\{0, 1, \ldots, n-1\}$ such that the sum of the elements

of S gives a remainder of j upon division by n. (By convention, the sum of the elements of the empty set is 0.) Prove or disprove:

$$f_n(j) \leq f_n(0),$$

for all $n \geq 1$ and all $0 \leq j < n$.

6. Let P be the set of all real polynomials all of whose coefficients are either 0 or 1. Find

$$\inf\{\alpha \in \mathbb{R} : \exists f \in P \text{ such that } f(\alpha) = 0\}$$

and

$$\sup\{\alpha \in \mathbb{R} : \exists f \in P \text{ such that } f(0) = 1 \text{ and } f(\alpha) = 0\}.$$

Here inf denotes infinum (greatest lower bound) and sup denotes supremum (least upper bound).

Somewhat more difficult:

$$\sup\{\alpha \in \mathbb{R} : \exists f \in P \text{ such that } f(i\alpha) = 0\},$$

where $i^2 = -1$.

7. Let n be a positive integer, and let X_n be the set of all $n \times n$ matrices whose entries are $+1$ or -1. Call a nonempty subset S of X_n *full* if whenever $A \in S$, then any matrix obtained from A by multiplying a row or column by -1 also belongs to S. Let $w(A)$ denote the number of entries of A equal to 1. Find, as a function of n,

$$\max_S \frac{1}{|S|} \sum_{A \in S} w(A)^3,$$

where S ranges over all full subsets of X_n. (Express your answer as a polynomial in n.)

8.* Let R be the region consisting of all triples (x, y, z) of nonnegative real numbers satisfying $x + y + z \leq 1$. Let $w = 1 - x - y - z$. Express the value of the triple integral

$$\iiint_R x^1\, y^9\, z^8\, w^4\, dx\, dy\, dz$$

in the form $a!\, b!\, c!\, d!/n!$, where a, b, c, d, and n are positive integers.

9.* Let n be a positive integer, and let $f(n)$ denote the last nonzero digit in the decimal expansion of $n!$. For instance, $f(5) = 2$.

(a) Show that if $a_1, a_2, \ldots, a_k$ are distinct positive integers, then $f(5^{a_1} + 5^{a_2} + \cdots + 5^{a_k})$ depends only on the sum $a_1 + a_2 + \cdots + a_k$.

(b) Assuming (a), we can define $g(s) = f(5^{a_1} + 5^{a_2} + \cdots + 5^{a_k})$, where $s = a_1 + a_2 + \cdots + a_k$. Find the least positive integer p for which

$$g(s) = g(s + p), \text{ for all } s \geq 1,$$

or else show that no such p exists.

10.* A *transversal* of an $n \times n$ matrix is a set of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

(a) Each entry of A is either -1, 1, or 0.

(b) All transversals of A have the same sum of their elements.

Find a formula for $f(n)$ of the form

$$a_1 \cdot b_1^n + a_2 \cdot b_2^n + a_3 \cdot b_3^n + a_4,$$

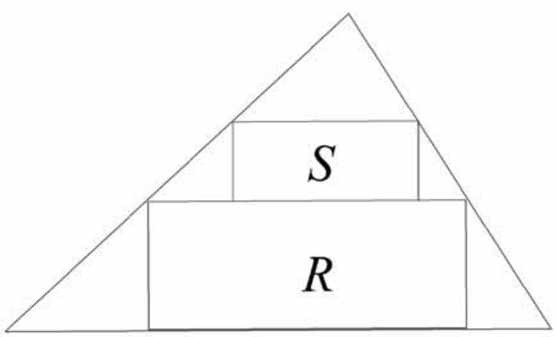
where a_i, b_i are rational numbers.

Easier version (not on Putnam Exam):

(a) Each entry of A is either 0 or 1.

(b) All transversals of A have the same number of 1's.

11.* Let T be a triangle and R, S rectangles inscribed in T as shown:

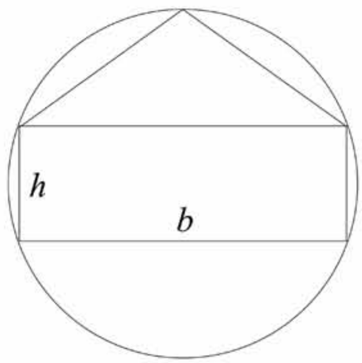


Find the maximum value, or show that no maximum exists, of

$$\frac{A(R) + A(S)}{A(T)},$$

where T ranges over all triangles and R, S over all rectangles as above, and where A denotes area.

12.* (A1 or B1 problem) Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown.



For what value of h do the rectangle and triangle have the same area?

13.* If $p(x) = \sum_{i=0}^{m} a_i x^i$ is a polynomial with real coefficients a_i, then set

$$,\ (p(x)) = \sum_{i=0}^{m} a_i^2.$$

Let $f(x) = 3x^2 + 7x + 2$. Find (with proof) a polynomial $g(x)$ satisfying

$$g(0) = 1, \text{ and}$$

$$,\ (f(x)^n) = ,\ (g(x)^n) \text{ for every integer } n \geq 1.$$

14.* Define polynomials $f_n(x)$ for $n \geq 0$ by

$$\begin{aligned} f_0(x) &= 1 \\ f'_{n+1}(x) &= (n+1) f_n(x+1),\ n \geq 0 \\ f_n(0) &= 0,\ n \geq 1. \end{aligned}$$

Find (with proof) the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

Variation (not on Putnam Exam): $f_0(x) = 1$, $f_{n+1}(x) = xf_n(x) + f_n'(x)$. Find $f_{2n}(0)$.

15. Define

$$c(k,n) = \cos\frac{\pi k}{n} + \sqrt{1 + \cos^2\frac{\pi k}{n}}.$$

Find (with proof) all positive integers n satisfying

$$c(1,n) = c(2,n)c(3,n).$$

16. Let R be a ring (not necessarily with identity). Suppose that there exists a nonzero element x of R satisfying

$$x^4 + x = 2x^3.$$

Prove or disprove: There exists a nonzero element y of R satisfying $y^2 = y$.

17. Find the largest real number λ for which there exists a 10×10 matrix $A = (a_{ij})$, with each entry a_{ij} equal to 0 or 1, and with exactly 84 0's, and there exists a nonzero column vector x of length 10 with real entries, such that $Ax = \lambda x$.

18. Choose two points p and q independently and uniformly from the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ in the (x,y)-plane. What is the probability that there exists a circle C contained entirely within the first quadrant $x \geq 0$, $y \geq 0$ such that C contains x and y in its interior? Express your answer in the form $1 - (a + b\pi)(c + d\sqrt{e})$ for rational numbers a, b, c, d, e.

19.* (A1 or B1 problem) Let k be the smallest positive integer with the following property:

> There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial $p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$ has exactly k nonzero coefficients.

Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

NOTE. The original version of this problem was considerably more difficult (and was not intended for A1 or B1). It was as follows:

Let $P(x) = x^{11} + a_{10}x^{10} + \cdots + a_0$ be a monic polynomial of degree eleven with real coefficients a_i, with $a_0 \neq 0$. Suppose that all the zeros of $P(x)$ are real, i.e., if α is a complex number such that $P(\alpha) = 0$, then α is real. Find (with proof) the least possible number of nonzero coefficients of $P(x)$ (including the coefficient 1 of x^{11}).

20. Find (with proof) the largest integer k for which there exist three 9-element subsets X_1, X_2, X_3 of real numbers and k triples (a_1, a_2, a_3) satisfying $a_i \in X_i$ and $a_1 + a_2 + a_3 = 0$.

21. Let
$$S = \sum \frac{1}{m^2n^2},$$
where the sum ranges over all pairs (m, n) of positive integers such that the largest power of 2 dividing m is different from the largest power of 2 dividing n. Express S in the form $\alpha\pi^k$, where k is an integer and α is rational. You may assume the formula
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

22. Let a and b be nonnegative integers with binary expansions $a = a_0 + 2a_1 + \cdots$ and $b = b_0 + 2b_1 + \cdots$ (so $a_i, b_i = 0$ or 1), and define
$$a \wedge b = a_0b_0 + 2a_1b_1 + 4a_2b_2 + \cdots = \sum 2^i a_i b_i.$$
Given an integer $n \geq 0$, define $f(n)$ to be the number of pairs (a, b) of nonnegative integers satisfying $n = a + b + (a \wedge b)$. Find a polynomial $P(x)$ for which
$$\sum_{n=0}^{\infty} f(n)x^n = \prod_{k=0}^{\infty} P\left(x^{2^k}\right), \quad |x| < 1,$$
or show that no such $P(x)$ exists.

23. Given $v = (v_1, \ldots, v_n)$ where each $v_i = 0$ or 1, let $f(v)$ be the number of even numbers among the n numbers

$$v_1+v_2+v_3, v_2+v_3+v_4, \ldots, v_{n-2}+v_{n-1}+v_n, v_{n-1}+v_n+v_1, v_n+v_1+v_2.$$

For which positive integers n is the following true: for all $0 \le k \le n$, exactly $\binom{n}{k}$ vectors of the 2^n vectors $v \in \{0,1\}^n$ satisfy $f(v) = k$?

24. Let p be a prime number. Let c_k denote the coefficient of x^{2k} in the polynomial $(1+x+x^3+x^4)^k$. Find the remainder when the number $\sum_{k=0}^{p-1}(-1)^k c_k$ is divided by p. Your answer should depend only on the remainder obtained when p is divided by some fixed number n (independent of p).

25. Let $x(t)$ and $y(t)$ be real-valued functions of the real variable t satisfying the differential equations

$$\begin{aligned} \frac{dx}{dt} &= -xt + 3xy - 2t^2 + 1 \\ \frac{dy}{dt} &= xt + yt + 2t^2 - 1, \end{aligned}$$

with the initial conditions $x(0) = y(0) = 1$. Find $x(1) + 3y(1)$. (This problem was later withdrawn for having an easier than intended solution.)

26.* Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be real numbers with $1 \le b_1 < b_2 < \cdots < b_n$. Suppose that there is a polynomial $f(x)$ satisfying

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Express $f(1)$ in terms of $b_1, \ldots, b_n$ and n (but independent from $a_1, \ldots, a_n$).

27. Given positive integers n and i, let x be the unique real number $\ge i$ satisfying $x^{x-i} = n$. Define $f(n,i) = (x+1)^{x-i}$, and set $f(0,i) = 0$ for all i. Suppose that $a_1, a_2, \ldots$ is a nonnegative integer sequence satisfying $a_{i+1} \le f(a_i, i)$ for all $i \ge 1$. Prove or disprove: a_i is a polynomial function of i for i sufficiently large.

28. Let $0 \le x \le 1$. Let the binary expansion of x be

$$x = a_1 2^{-1} + a_2 2^{-2} + \cdots$$

(where, say, we never choose the expansion ending in infinitely many 1's). Define

$$f(x) = a_1 3^{-1} + a_2 3^{-2} + \cdots.$$

In other words, write x in binary and read x in ternary. Evaluate $\int_0^1 f(x)dx$.

29.* Let $f(x, y, z) = x^2 + y^2 + z^2 + xyz$. Let $p(x, y, z)$, $q(x, y, z)$, and $r(x, y, z)$ be polynomials satisfying

$$f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).$$

Prove or disprove: (p, q, r) consists of some permutation of $(\pm x, \pm y, \pm z)$, where the number of minus signs is even.

30. Let

$$\frac{1}{1 - x - y - z - 6(xy + xz + yz)} = \sum_{r,s,t=0}^{\infty} f(r, s, t) x^r y^s z^t$$

(convergent for $|x|, |y|, |z|$ sufficiently small). Find the largest real number R for which the power series

$$F(u) = \sum_{n=0}^{\infty} f(n, n, n) u^n$$

converges for all $|u| < R$.