18.S34 (FALL 2006)

PROBLEMS ON INEQUALITIES

1. Let a be a real number and n a positive integer, with $a > 1$. Show that

$$a^n - 1 \geq n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right).$$

2. Let $x_i > 0$ for $i = 1, 2, \ldots, n$. Show that

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2.$$

3. If $x_i > 0$, $q_i > 0$ for $i = 1, 2, \ldots, n$, and $q_1 + \cdots + q_n = 1$, show that

$$x_1^{q_1} \cdots x_n^{q_n} \leq q_1 x_1 + \cdots + q_n x_n.$$

4. For $p > 1$ and $a_1, a_2, \ldots, a_n$ positive, show that

$$\sum_{k=1}^{n} \left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{n} a_k^p.$$

5. If $a_n > 0$ for $n = 1, 2, \ldots$, show that

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 a_2 \cdots a_n} \leq e \sum_{n=1}^{\infty} a_n,$$

provided that $\sum_{n=1}^{\infty} a_n$ converges.

6. Let $0 < x < \pi/2$. Show that

$$x - \sin x \leq \frac{1}{6}x^3.$$

7. Show that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2.$$

8. Let
$$\frac{a_1}{b_1},\ \frac{a_2}{b_2},\ \ldots,\ \frac{a_n}{b_n}$$
be n fractions with $b_i > 0$ for $i = 1, 2 \ldots, n$. Show that the fraction
$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}$$
is contained between the largest and smallest of these n fractions.

9. For $n = 1, 2, 3, \ldots$ let
$$x_n = \frac{1000^n}{n!}.$$
Find the largest term of the sequence.

10. Suppose that $a_1, a_2, \ldots, a_n$ with $n \geq 2$ are real numbers larger than -1, and moreover all a_j's have the same sign. Show that
$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) > 1 + a_1 + a_2 + \cdots + a_n.$$

11. Show that
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}.$$

12. Prove *Chebyshev's inequality*: If $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$, then
$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \leq \frac{1}{n}\sum_{k=1}^{n} a_k b_k.$$
Generalize to more than two sets of increasing sequences.

13. Let n be a positive integer larger than 1, and let $a > 0$. Show that
$$\frac{1 + a + a^2 + \cdots + a^n}{a + a^2 + a^3 + \cdots + a^{n-1}} \geq \frac{n+1}{n-1}.$$

14. Show that if $a > b > 0$, then $A < B$, where
$$A = \frac{1 + a + \cdots + a^{n-1}}{1 + a + \cdots + a^n}, \quad B = \frac{1 + b + \cdots + b^{n-1}}{1 + b + \cdots + b^n}.$$

15. Let $x > 0$, and let n be a positive integer. Show that

$$\frac{x^n}{1 + x + x^2 + \cdots + x^{2n}} \leq \frac{1}{2n+1}.$$

16. Let $a, b > 0,\ a + b = 1$, and $q > 0$. Show that

$$\left(a + \frac{1}{a}\right)^q + \left(b + \frac{1}{b}\right)^q \geq \frac{5^q}{2^{q-1}}.$$

17. Let $x, y > 0$ with $x \neq y$, and let m and n be positive integers. Show that

$$x^m y^n + x^n y^m < x^{m+n} + y^{m+n}.$$

18. Let $x > 0$ but $x \neq 1$, and let n be a positive integer. Show that

$$x^{2n-1} + x < x^{2n} + 1.$$

19. Let $a > b > 0$, and let n be a positive integer greater than 1. Show that

$$\sqrt[n]{a} - \sqrt[n]{b} < \sqrt[n]{a-b}.$$

20. Let $a, b, x > 0$ and $a \neq b$. Show that

$$\left(\frac{a+x}{b+x}\right)^{b+x} > \left(\frac{a}{b}\right)^x.$$

21. Let $a > b > 0$, and let n be a positive integer greater than 1. Show that for $k \geq 0$,

$$\sqrt[n]{a^n + k^n} - \sqrt[n]{b^n + k^n} \leq a - b.$$

22. Let $x \geq 0$, and let m and n be real numbers such that $m \geq n > 0$. Show that

$$(m+n)(1 + x^m) \geq 2n\frac{1 - x^{m+n}}{1 - x^n}.$$

23. Let $a_i \geq 0$ for $1 \leq i \leq n$, and let $\sum_{i=1}^{n} a_i = 1$. Let $0 \leq x_i \leq 1$ for $1 \leq i \leq n$. Show that

$$\frac{a_1}{1+x_1} + \frac{a_2}{1+x_2} + \cdots + \frac{a_n}{1+x_n} \leq \frac{1}{1 + x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}.$$

24. If $a_1, \ldots, a_{n+1}$ are positive real numbers with $a_1 = a_{n+1}$, show that

$$\sum_{i=1}^{n} \left(\frac{a_i}{a_{i+1}}\right)^n \geq \sum_{i=1}^{n} \frac{a_{i+1}}{a_i}.$$

25. Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be two sets of real numbers with $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$. Put $s_k = a_1 + a_2 + \cdots + a_k$ for $k = 1, 2, \ldots, n$; and let M and m denote respectively the largest and smallest of the numbers $s_1, s_2, \ldots, s_n$. Show that

$$mb_1 \leq \sum_{i=1}^{n} a_i b_i \leq M b_1.$$

26. Show that for any real numbers $a_1, a_2, \ldots, a_n$,

$$\left(\sum_{i=1}^{n} \frac{a_i}{i}\right)^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j}{i+j-1}.$$

27. Let f and g be real-valued functions defined on the set of real numbers. Show that there are numbers x and y such that $0 \leq x \leq 1,\ 0 \leq y \leq 1$, and

$$|xy - f(x) - g(x)| \geq 1/4.$$

28. Let $t > 0$. Show that

$$t^\alpha - \alpha t \leq 1 - \alpha, \quad \text{if } 0 < \alpha < 1$$

and

$$t^\alpha - \alpha t \geq 1 - \alpha, \quad \text{if } \alpha > 1.$$

29. Show that for any real number x and any positive integer n we have

$$\left|\sum_{k=1}^{n} \frac{\sin kx}{k}\right| \leq 2\sqrt{\pi}.$$

30. Show that if x is larger than any of the numbers $a_1, a_2, \ldots, a_n$, then

$$\frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_n} \geq \frac{n}{x - \frac{1}{n}(a_1 + a_2 + \cdots + a_n)}.$$

31. Show that

$$\sqrt{\binom{n}{1}} + \sqrt{\binom{n}{2}} + \cdots + \sqrt{\binom{n}{n}} \leq \sqrt{n(2^n - 1)}.$$

32. Let $y = f(x)$ be a continuous, strictly increasing function of x for $x \geq 0$, with $f(0) = 0$, and let f^{-1} denote the inverse function to f. If a and b are nonnegative constants, then show that

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy.$$

33. Show that for $t \geq 1$ and $s \geq 0$,

$$ts \leq t \log t - t + e^s.$$

34. Let $a_1/b_1,\ a_2/b_2,\ \ldots$, with each $b_i > 0$, be a strictly increasing sequence. Let

$$A_j = a_1 + a_2 + \cdots + a_j, \quad and \quad B_j = b_1 + b_2 + \cdots + b_j.$$

Show that the sequence $A_1/B_1,\ A_2/B_2,\ \ldots$ is also strictly increasing.

35. Let $m,\ n$ be positive integers, and let $a_1, a_2, \ldots, a_n$ be positive real numbers. For $i = 1, 2, 3 \ldots$ put $a_{n+i} = a_i$ and

$$b_i = a_{i+1} + a_{i+2} + \cdots + a_{i+m}.$$

Show that

$$m^n a_1 a_2 \cdots a_n < b_1 b_2 \cdots b_n,$$

except if all the a_i are equal.

36. Let $a_1, a_2, \ldots, a_n$ be real numbers. Show that

$$\min_{a_i \neq a_j} (a_i - a_j)^2 \leq M^2 \left(a_1^2 + \cdots + a_n^2\right),$$

where

$$M^2 = \frac{12}{n(n^2 - 1)}.$$

37. Let x and a be real numbers, and let n be a nonnegative integer. Show that
$$|x-a|^n|x+na| \le \left(x^2+na^2\right)^{(n+1)/2}.$$

38. Given an arbitrary finite set of n pairs of positive real numbers $\{(a_i,b_i) : i=1,2,\ldots,n\}$, show that
$$\prod_{i=1}^{n}(xa_i+(1-x)b_i) \le \max\left\{\prod_{i=1}^{n}a_i,\ \prod_{i=1}^{n}b_i\right\},$$
for all $x\in[0,1]$. Equality is attained only at $x=0$ or $x=1$, and then if and only if
$$\left(\sum_{i=1}^{n}\frac{a_i-b_i}{a_i}\right)\left(\sum_{i=1}^{n}\frac{a_i-b_i}{b_i}\right) \ge 0.$$

39. Show that if m and n are positive integers, then the smallest of the numbers $\sqrt[n]{m}$ and $\sqrt[m]{n}$ cannot exceed $\sqrt[3]{3}$.

40. Show that if $a\ge 2$ and $x>0$, then $a^x+a^{1/x}\le a^{x+1/x}$, with equality holding if and only if $a=2$ and $x=1$.

41. Show that if $x_i \ge 0$ for $i=1,2\ldots,n$ and $\sum_{i=1}^{n}\frac{1}{1+x_i} \le 1$, then $\sum_{i=1}^{n}2^{-x_i}\le 1$.

42. Let $0\le a_i<1$ for $i=1,2,\ldots,n$, and put $\sum_{i=1}^{n}a_i=A$. Show that
$$\sum_{i=1}^{n}\frac{a_i}{1-a_i}\ge\frac{nA}{n-A},$$
with equality if and only if all the a_i are equal.

43. Show that for $n\ge 2$,
$$\prod_{i=0}^{n}\binom{n}{i}\le\left(\frac{2^n-2}{n-1}\right)^{n-1}.$$

44. Let $b_1,\ldots,b_n$ be any rearrangement of the positive numbers $a_1,\ldots,a_n$. Show that
$$\frac{a_1}{b_1}+\cdots+\frac{a_n}{b_n}\ge n.$$

45. Given that $\sum_{i=1}^{n} b_i = b$ with each b_i a nonnegative number, show that

$$\sum_{j=1}^{n-1} b_j b_{j+1} \le \frac{b^2}{4}.$$

46. Let $n \ge 2$ and $0 < x_1 < x_2 < \cdots < x_n \le 1$. Show that

$$\frac{n \displaystyle\sum_{k=1}^{n} x_k}{\displaystyle\sum_{k=1}^{n} x_k + n x_1 x_2 \cdots x_n} \ge \sum_{k=1}^{n} \frac{1}{1+x_k}.$$

47. Let f be a continuous function on the interval $[0,1]$ such that $0 < m \le f(x) \le M$ for all x in $[0,1]$. Show that

$$\left(\int_0^1 \frac{dx}{f(x)}\right)\left(\int_0^1 f(x)dx\right) \le \frac{(m+M)^2}{4mM}.$$

48. Let $x > 0$ and $x \ne 1$. Show that

$$\frac{\log x}{x-1} \le \frac{1}{\sqrt{x}}$$
$$\frac{\log x}{x-1} \le \frac{1+x^{1/3}}{x+x^{1/3}}.$$

49. Let $0 < y < x$. Show that

$$\frac{x+y}{2} > \frac{x-y}{\log x - \log y}.$$

50. Let $x > 0$. Show that

$$\frac{2}{2x+1} < \log\left(x + \frac{1}{x}\right) < \frac{1}{\sqrt{x^2+x}}.$$

51. Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Show that

$$n\left\{(1+n)^{1/n} - 1\right\} < S_n < n\left\{1 - (n+1)^{-1/n} - \frac{1}{n+1}\right\}.$$

52. Let $x > 0$ and $y > 0$. Show that

$$\frac{1-e^{-x-y}}{(x+y)(1-e^{-x})(1-e^{-y})} - \frac{1}{xy} \le \frac{1}{12}.$$

53. Let $a,\ b,\ c,\ d,\ e$, and f be nonegative real numbers satisfying

$$a+b \le e \quad \text{and} \quad c+d \le f.$$

Show that

$$\sqrt{ac}+\sqrt{bd} \le \sqrt{ef}.$$

54. Show that for $x > 0$ and $x \ne 1$,

$$0 \le \frac{x \log x}{x^2-1} \le \frac{1}{2}.$$

55. Show that for $x > 0$,

$$x(2+\cos x) > 3 \sin x.$$

56. Show that for $0 < x < \pi/2$,

$$2 \sin x + \tan x > 3x.$$

57. Let $x > 0,\ x \ne 1$, and suppose that n is a positive integer. Show that

$$x + \frac{1}{x^n} > 2n \frac{x-1}{x^n-1}.$$

58. Let a be a fixed real number such that $0 \le a < 1$, and let k be a positive integer satisfying the condition $k > (3+a)/(1-a)$. Show that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{nk-1} > 1 + a$$

for any positive integer n.

59. Let a and b denote real numbers, and let r satisfy $r \ge 0$. Show that

$$|a+b|^r \le c_r \left(|a|^r + |b|^r\right),$$

where $c_r = 1$ for $r \le 1$ and $c_r = 2^{r-1}$ for $r > 1$.

60. Let $0 < b \le a$. Show that

$$\frac{1}{8}\frac{(a-b)^2}{a} \le \frac{a+b}{2} - \sqrt{ab} \le \frac{1}{8}\frac{(a-b)^2}{b}.$$

61. Consider any sequence $a_1, a_2, \ldots$ of real numbers. Show that

$$\sum_{n=1}^{\infty} a_n \le \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \left(\frac{r_n}{n}\right)^{1/2}, \tag{1}$$

where

$$r_n = \sum_{k=n}^{\infty} a_k^2.$$

(If the left-hand side of (1) is ∞, then so is the right-hand side.)

62. Let a, b, and x be real numbers such that $0 < a < b$ and $0 < x < 1$. Show that

$$\left(\frac{1-x^b}{1-x^{a+b}}\right)^b > \left(\frac{1-x^a}{1-x^{a+b}}\right)^a.$$

63. Let $0 < a < 1$. Show that

$$\frac{2}{e} < a^{\frac{a}{1-a}} + a^{\frac{1}{1-a}} < 1.$$

64. Let $0 < x < 2\pi$. Show that

$$-\frac{1}{2}\tan\frac{x}{4} \le \sum_{k=1}^{n} \sin kx \le \frac{1}{2}\cot\frac{x}{4}.$$

65. Let $0 < a_k < 1$ for $k = 1, 2, \ldots, n$, with $a_1 + a_2 + \cdots + a_n < 1$. Show that

$$\frac{1}{1 - \sum_{k=1}^{n} a_k} > \prod_{k=1}^{n}(1 + a_k) > 1 + \sum_{k=1}^{n} a_k$$

and

$$\frac{1}{1 + \sum_{k=1}^{n} a_k} > \prod_{k=1}^{n}(1 - a_k) > 1 - \sum_{k=1}^{n} a_k$$

66. Show that

$$\frac{1}{(n-1)!}\int_n^{\infty} w(t)e^{-t}dt < \frac{1}{(e-1)^n},$$

where t is real, n is a positive integer, and

$$w(t) = (t-1)(t-2)\cdots(t-n+1).$$