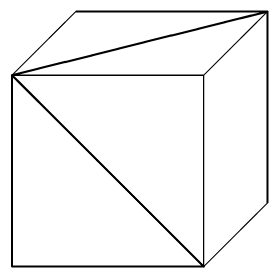
18.S34 PROBLEMS #9

Fall 2007

96. [1] Find the size of the planar angle formed by two face diagonals of a cube with a common vertex. (Try to find an elegant, noncomputational solution.)



97. [1.5] Find the missing term:

$$10, 11, 12, 13, 14, 15, 16, 17, 20, 22, 24, 31, 100, \_\_\_\_, 10000.$$

98. [1.5] Explain the rule which generates the following sequence:

$$2, 3, 10, 12, 13, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30,$$
$$31, 32, 33, 34, 35, 36, 37, 38, 39, 200, 201, 202, \ldots$$

HINT: *Don't* think mathematically!

99. [1] (a) Two players play the following game. They start with a pile of 101 stones. The players take turns removing either 1, 2, 3, or 4 stones from the pile. The player who takes the last stone wins. Assuming both players play perfectly, will the first or second player win?

    (b) What if the person who takes the last stone loses?

100. [1.5] Why does a mirror reverse left and right but not up and down? (This is not a frivolous question.)

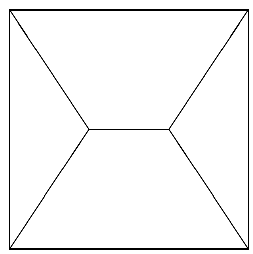
101. [1] Solve the recurrence

$$f(n+1) = nf(n) + (n-1)f(n-1) + \cdots + 2f(2) + f(1) + 1, \qquad f(0) = 1.$$

102. [2] From a 100×100 chessboard remove any white square and any black square. Show that the remaining board can be covered with 4999 non-overlapping dominoes. (Each domino covers two adjacent squares.)

103. A collection of line segments inside or on the boundary of a square of side one is said to be *opaque* if every (infinite) straight line which crosses the square makes contact with at least one of the segments. For example, the two diagonals are opaque of total length $2\sqrt{2}\approx 2.82$.

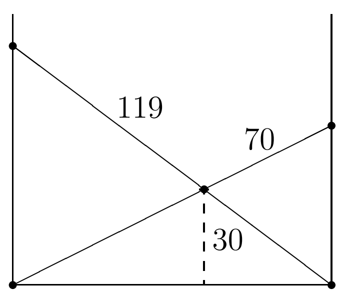
    (a) [2] The following symmetric pattern is opaque. (The sides of the square are not part of the pattern.)

    Show that its minimum total length is $1+\sqrt{3}\approx 2.73$.

    (b) [2.5] Can you find a shorter opaque set? So far as I know, the smallest known opaque set has length $\sqrt{2}+\frac{1}{2}\sqrt{6}\approx 2.64$, and it is not known whether a smaller one exists.

104. [2.5] Two ladders of length 119 feet and 70 feet lean between two vertical walls so that they cross 30 feet above the ground. How far apart are the walls?



105. [3] Let G denote the set of all infinite sequences $(a_1,a_2,\ldots)$ of integers a_i. We can add elements of G coordinate-wise, i.e.,

$$(a_1,a_2,\ldots)+(b_1,b_2,\ldots)=(a_1+b_1,a_2+b_2,\ldots).$$

Let $\mathbb{Z}$ denote the set of integers. Suppose $f : G \to \mathbb{Z}$ is a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in G$.

(a) Let e_i be the element of G with a 1 in position i and 0's elsewhere. Suppose that $f(e_i) = 0$ for all i. Show that $f(x) = 0$ for all $x \in G$. (Note. From the fact that f preserves the sum of two elements it follows easily that f preserves *finite* sums. However, it does not necessarily follow that f preserves *infinite* sums.)

(b) Show that $f(e_i) = 0$ for all but finitely many i.

106. [5] Let $\left(\frac{n}{7}\right)$ denote the Legendre symbol. Specifically,

$$\left(\frac{n}{7}\right) = \begin{cases} 0, & n \equiv 0 \pmod 7 \\ 1, & n \equiv 1, 2, 4 \pmod 7 \\ -1, & n \equiv 3, 5, 6 \pmod 7. \end{cases}$$

Show that

$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt = \sum_{n \geq 1} \left(\frac{n}{7}\right) \frac{1}{n^2}.$$

The main point of this exercise is to give an example of an explicit identity which is only conjecturally true.