18.S096 Problem Set 1 Fall 2013

Due date **: 9/24/2013**
Collaboration on homework is encouraged, but you should think through the problems yourself before discussing them with other people. **You must write your solution in your own words. Make sure to list all your collaborators.**

Part A

Part A has problems that straightforwardly follow from the definition. Use this part as an opportunity to get used to the concepts and definitions.

Problem A-1. Decide whether the following statements are true or false. No explanation needed.
(a) The row-rank and column-rank of a matrix is always equal to each other.
(b) For a $m \times n$ matrix A and a $n \times \ell$ matrix B, it is always true that $rank(AB) \leq rank(A)$.
(c) There are matrices whose left-inverse and right-inverse are not equal to each other, i.e., $\exists A, B, C$ such that $BA = I$ and $AC = I$.
(d) A matrix has full-rank if and only if it is invertible.
(e) All symmetric matrices are invertible.
(f) A matrix is invertible if all eigenvalues are distinct.
(g) A matrix is diagonalizable if and only if it is invertible.

Problem A-2. Compute the row-rank and column-rank of the following matrices.

$$(a) \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \qquad (b) \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 4 & 0 & -2 & 0 \end{pmatrix}$$

Problem A-3. Compute the determinant and inverse of the following matrices.

$$(a) \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix} \qquad (b) \begin{pmatrix} -1 & -2 & 3 \\ 1 & 2 & 0 \\ 4 & 6 & 3 \end{pmatrix}.$$

Problem A-4. Compute the charactersitc polynomial and find the eigenvalues and eigenvectors of the following matrix:

$$\begin{pmatrix} -3 & 3 & 2 \\ 1 & -1 & -2 \\ -1 & -3 & 0 \end{pmatrix}.$$

Problem A-5. (a) Find an orthonormal basis of the subspace of $\mathbb{R}^5$ spanned by v_1, v_2, and v_3, where $v_1 = (1,0,1,0,1)$, $v_2 = (1,1,1,0,0)$, and $v_3 = (0,0,1,1,1)$, using Gram-Schmidt process.

(b) Complete the basis found in (a), into an orthonormal basis of $\mathbb{R}^5$.
(c) Find the matrix U which transforms the basis found in (b) into the standard basis of $\mathbb{R}^5$.

Problem A-6. Use the singular-value decomposition to express the following matrices in the form $A = U\Sigma V^T$.

$$(a)\begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \qquad (b)\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \qquad (c)\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Part B

Part B has more elaborate problems. Many of the problems in Part B cover important topics that we did not have enough time to cover in lecture. Thus understanding the content is as important as solving the problem. Try to think through the content of the problem while solving it.

Problem B-1. Decide whether the following statements about a singular value decomposition $A = U\Sigma V^T$ are true or false. EXPLAIN the reason.
(a) The number of non-zero diagonal entries of Σ is equal to the rank of A.
(b) Σ is uniquely determined up to permuting the rows and columns.
(c) If A is a $n \times n$ matrix, then U and V can be chosen so that $U = V$.
(d) If A is a symmetric $n \times n$ matrix, then U and V can be chosen so that $U = V$.

Problem B-2. Prove that the matrices AA^T and A^TA have the same set of non-zero eigenvalues. (Hint: consider the singular value decomposition of A)

Problem B-3. A matrix A is *positive semi-definite* if

$$v^T A v \geq 0$$

for all non-zero vectors v. Prove that for all symmetric positive semi-definite matrices A, there exists an orthonormal matrix U such that

$$D = U^T A U$$

is a diagonal matrix such that all diagonal entries are non-negative.

Problem B-4. (Linear regression) Suppose that a set of n vectors $v_1, v_2, \cdots, v_n \in \mathbb{R}^m$ and a vector $w \in \mathbb{R}^m$ is given. We seek reals $x_1, x_2, \cdots, x_n$ that minimizes the following quantity:

$$L = ||(\sum_{i=1}^{n} x_i v_i) - w||.$$

Let A be an $m \times n$ matrix whose i-th column is the vector v_i and let $\vec{x}$ be an n-dimensional column vector whose i-th coordinate is x_i. Then the above can be re-wrtten as minimizing

$$L = ||A\vec{x} - w||.$$

(a) Show that the problem can be directly solved if A is a diagonal matrix.
(b) Suppose that the SVD of A is given as $A = U\Sigma V^T$. Then

$$A\vec{x} - w = U\Sigma V^T\vec{x} - w = U\Sigma V^T\vec{x} - UU^T w = U(\Sigma V^T\vec{x} - U^T w).$$

Use this equation to explain how the general problem reduces to the special case when A is a diagonal matrix.

(c) Suppose that we are given n data points $(x_1, y_1), \cdots, (x_n, y_n)$ in $\mathbb{R}^2$. Our goal is to find a linear equation $y = ax + b$ that 'best fits' the data. Formally, we would like to find reals a and b that minimizes

$$\sum_{i=1}^{n}(ax_i + b - y_i)^2.$$

Explain how this problem is a special case of the problem given above.

(d) Extend (c) to describe the problem of finding a quadratic equation $y = ax^2 + bx + c$ that 'best fits' the data points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$ as a special case of the problem given above.

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