18.997 Topics in Combinatorial Optimization	18 March 2004
Lecture 12	
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Last time, we stated the following theorem by Edmonds and Lawler about the maximum independent set common to two matroids.

Theorem 1 *Let $\mathcal{M}_1 = (S, \mathcal{I}_1)$ and $\mathcal{M}_2 = (S, \mathcal{I}_2)$ be two matroids on common ground set S with rank functions r_1 and r_2, then*

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \in S} \left(r_1(U) + r_2(S \setminus U) \right).$$

To prepare for the proof, we proved some lemmas and stated the following algorithm. For $I \subseteq S$, let the digraph $D(I) = (S, A)$ be defined as follows: for $y \in I$, $x \notin I$, we have an arc $(y, x) \in A$ if $I - y + x \in \mathcal{I}_1$ and $(x, y) \in A$ if $I - y + x \in \mathcal{I}_2$. This is the union of the arcset $A_{\mathcal{M}_1}(I)$ corresponding to $\mathcal{I}_1$ and the reverse of the arcset $A_{\mathcal{M}_2}(I)$ corresponding to $\mathcal{I}_2$. Consider the sets

$$X_1 = \{x \in S \setminus I : I + x \in \mathcal{I}_1\}, X_2 = \{x \in S \setminus I : I + x \in \mathcal{I}_2\}.$$

The algorithm is as follows:

Matroid Intersection Algorithm (MIA)

Input Matroids $\mathcal{M}_1 = (S, \mathcal{I}_1)$, $\mathcal{M}_2 = (S, \mathcal{I}_2)$

Output $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ of maximum size

$I \leftarrow \emptyset$

while $D(I)$ has a path from X_1 to X_2

$I \leftarrow I \Delta V(P)$, where P is a *shortest* path from X_1 to X_2.

The choice of a *shortest* path P is crucial; otherwise, the algorithm is not correct. The correctness of the algorithm is proved below.

Theorem 2 *In any step of Algorithm* MIA*, if there is no path from X_1 to X_2 then I is of maximum size. Otherwise, if P is a shortest path from X_1 to X_2 then $J = I \Delta V(P)$ is an independent set in $\mathcal{I}_1$ and $\mathcal{I}_2$.*

Proof: If there is no path from X_1 to X_2 then let U be the set of vertices that can reach X_2. By assumption, $U \cap X_1 = \emptyset$ (and $(S \setminus U) \cap X_2 = \emptyset$). We show that $r_1(U) = |I \cap U|$ and $r_2(S \setminus U) = |I \cap (S \setminus U)|$. For contradiction, assume that $|I \cap U| \neq r_1(U)$, then $|I \cap U| < r_1(U)$. Thus, there exists $x \in U \setminus I$ such that $(I \cap U) + x \in \mathcal{I}_1$, but we know that $I + x \notin \mathcal{I}_1$, since otherwise x would be both in X_1 and in U and there would be a path from X_1 to X_2. Since both $(I \cap U) + x$ and I are independent, we can repeatedly add elements of the latter to the former until we get an independent set of size $|I|$. Thus there exists a $y \in I \setminus U$ such that $I + x - y \in \mathcal{I}_1$. By definition, there is an edge from y to x in $D(I)$ and it contradicts the definition of U. Similarly, one can prove that $r_2(S \setminus U) = |I \cap (S \setminus U|)$. This shows that I is of maximum size since $|I| = |I \cap U| + |I \cap (S \setminus U)| = r_1(U) + r_2(S \setminus U)$.

To prove the second statement, let P be a shortest path from X_1 (say $x_1 \notin I$) to X_2 and $J = I \Delta V(P)$. We prove that $J \in \mathcal{I}_1$ and similarly one can prove $J \in \mathcal{I}_2$. We augment the matroid $\mathcal{M}$ to $\mathcal{M}' = (S + t, \{I' | I' \setminus \{t\} \in \mathcal{I}_1\})$. Now in $D_{\mathcal{M}'}(I')$, t is connected (only) to X_1 and $J \cup \{t\}$ has a unique matching. This matching comes from taking the arcs of P of $A_{\mathcal{M}_1}(I)$ and adding (t, x_1). The fact that it is unique comes from the fact that P is a shortest path; otherwise another matching would lead to a shortcut in P. Now, we use the following lemma that we proved last time,

Lemma 3 *Given matroid $\mathcal{M} = (S, \mathcal{I}), I \in \mathcal{I}$, and $J \subseteq S$ with $|I| = |J|$, if $A_{\mathcal{M}}(I)$ contains a* unique *matching on $I \Delta J$, then $J \in \mathcal{I}$.*

Using this lemma, $J \cup \{t\} \in \mathcal{I}(\mathcal{M}')$. Thus, $J \in \mathcal{I}_1$ as desired. Similarly, one can show that $J \in \mathcal{I}_2$. □

Note that in the proof of Theorem 2, we showed that at the end of the algorithm (when there is no path from X_1 to X_2), there exists a set U for which the equality of Theorem 1 holds. Thus, the proof of Theorem 2 also shows Theorem 1.

1 Intersection of Many Matroids

Despite intersection of two matroids, the problem of finding the independent set of maximum size in the intersection of three matroids is NP-Hard.

Theorem 4 *Given three matroids $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ where $\mathcal{M}_i = (S, \mathcal{I}_i)$, it is NP-hard to find the independent set I with maximum size in $\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$.*

Proof: The reduction is from the Hamiltonian path problem. Let $D = (V, E)$ be a directed graph and s and t are two vertices in D. Given an instance $(D = (V, E), s, t)$ of the Hamiltonian path problem, we construct three matroids as follows: $\mathcal{M}_1$ is equal to the graphic matroid of the undirected graph G which is the undirected version of D. $\mathcal{M}_2 = (E, \mathcal{I}_2)$ is a partition matroid in which a subset of edges in an independent set if each vertex has at most one incoming edge in this set, i.e, $\mathcal{I}_2 = \{F \subseteq E : |\delta^-(v) \cap F| \leq f_s(v)\}$ where $f_s(v) = 1$ if $v \neq s$ and $f_s(s) = 0$. Similarly, we define $\mathcal{M}_3 = (E, \mathcal{I}_3)$ such that $\mathcal{I}_3 = \{F \subseteq E : |\delta^+(v) \cap F| \leq f_t(v)\}$ where $f_t(v) = 1$ if $v \neq t$ and $f_t(t) = 0$. It is easy check that any set in the intersection of these matroids corresponds to the union of vertex-disjoint directed paths with one of them starting at s and one (possibly a different one) ending at t. Therefore, the size of this set is $n - 1$ if and only if there exists a Hamiltonian path from s to t in D. □

2 Maximum Weight Common Independent set of two matroids

We give an algorithm to find the maximum weight common independent set of two matroids. Here is a brief description of the algorithm. At step i of the algorithm, we find the maximum weight independent set of size i and at the end, we output the independent set of maximum weight among all of these independent sets.

We start from an empty set as I_0. Suppose I_i is a maximum weight common independent set of size i. Let $l(s) = w(s)$ if $s \in I_i$ and $l(s) = -w(s)$ if $s \notin I_i$. We find the maximum weight common independent set of size $i + 1$ by first constructing the digraph $D(I)$ as in the maximum cardinality matroid intersection algorithm and then by proceeding as follows:

1. If no path from X_1 to X_2 exists, then there is no larger common independent set
2. else find a path, P, from X_1 to X_2 of shortest total length $l(P)$ and if several paths have the same weighted length $l(P)$, we choose the path among them with minimum number of vertices. Then $I_{i+1} = I_i \Delta V(P)$.

The fact that we started from a maximum weight independent set of size i can be seen to imply that the weighted digraph we construct has no negative length directed cycles (and hence the computation of the shortest path P makes sense and can be done efficiently). For the proof of the correctness of this algorithm, we refer the reader to the textbook.

3 Matroid Intersection Polytope

Edmonds [1970] has characterized all inequalities defining the *matroid intersection polytope*, the convex hull of independent sets common to two matroids. In this lecture, we state the characterization of this polytope. In the next lecture, we prove its integrality by showing in a very elegant way that the corresponding system of linear inequalities is totally dual integral.

Given matroids $\mathcal{M}_1(S, \mathcal{I}_1)$ and $\mathcal{M}_2(S, \mathcal{I}_2)$, the matroid intersection polytope is the following:

$$\begin{aligned} x(U) &\leq r_1(U) \quad \forall U \subseteq S \\ x(U) &\leq r_2(U) \quad \forall U \subseteq S \\ x_s &\geq 0 \quad \forall s \in S \end{aligned}$$

where x_s is a variable for each element s of S; and $x(U) = \sum_{s \in U} x_s$.

4 Matroid Union

Given k matroids $\left(\mathcal{M}_i = (S_i, \mathcal{I}_i)|_{i=1}^k\right)$ on possibly different ground sets, it can be shown that the independence system $(\cup_{i=1}^k S_i, \{\cup_{i=1}^k I_i | I_i \in \mathcal{I}_i \text{ for } 1 \leq i \leq k\})$ is a matroid called the union matroid of $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ denoted by $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2 \vee \ldots \vee \mathcal{M}_k$. The rank function of matroid $\mathcal{M}$ is $r_\mathcal{M}(U) = \min_{T \subset U} \left[(U \setminus T) + \sum_{i=1}^k r_i(T \cap S_i)\right]$ for any $U \subset \cup_{i=1}^k S_i$ (the fact that the rank function is at most this quantity is very easy to see since $I \cap T \cap S_i$ over all i covers $I \cap T$). Next lecture, we will prove these facts about matroid union by deducing them from matroid intersection.

Let $\mathcal{M}^{(k)}$ be the union of k copies of matroid $\mathcal{M}$. By the above formula, we have $r_{\mathcal{M}^{(k)}}(U) = \min_{T \subseteq U}(|U \setminus T| + k r_\mathcal{M}(T))$. Thus, S has k disjoint bases if and only if $k r_\mathcal{M}(S) = \min_{T \subseteq U}(|U \setminus T| + k r_\mathcal{M}(T))$. This is equivalent to saying that for all $T \subseteq S$: $|S \setminus T| \geq k(r_\mathcal{M}(S) - r_\mathcal{M}(T))$. In addition, S can be covered by k independent sets if and only if for all $T \subseteq S$: $|T| \leq k r_\mathcal{M}(T)$. Nash-Williams and Tutte-Nash-Williams theorems in graphs are corollaries of these facts. See Lecture 14 for precise statement and proofs.

5 Shannon Switching Game

Here, we state the generalization of the two-player game from Lecture 11 (on general matroids) and show the winning strategy in these games. The game is played on a matroid $\mathcal{M} = (S, \mathcal{I})$. Player 2's moves consist of fixing an element of S and player 1's moves consist of deleting any unfixed element in S. The game ends when every element has been fixed or deleted. Player 1 plays first. Player 2 wins if he can fix a basis of the matroid. Otherwise player 1 wins. The question is to find the winning strategy of this game.

First, note that given a matroid $\mathcal{M}$, either player 1 or 2 has a winning strategy. So the problem is to characterize the set of all graphs for which player 2 has a winning strategy.

Theorem 5 *Player 2 has a winning strategy if and only if S has two disjoint bases.*

Proof:

Case 1: If S does not have two disjoint bases then, from the results above regarding the union of two identical matroids, we derive that there exists a subset $T \subseteq S$ such that $|S \setminus T| < 2(r_\mathcal{M}(S) - r_\mathcal{M}(T))$. Now the strategy of player 1 is to always delete an element from $S \setminus T$. Therefore, player 1 can delete at least $\left\lceil \frac{|S \setminus T|}{2} \right\rceil$ of elements. Hence, player 2 can fix at most $\left\lfloor \frac{|S \setminus T|}{2} \right\rfloor < r_\mathcal{M}(S) - r_\mathcal{M}(T)$ elements within $S \setminus T$ and $r_\mathcal{M}(T)$ elements within T. Thus, player 2 can fix less than $r_\mathcal{M}(S)$ elements and will not be able to fix a basis.

Case 2: In this case, $\mathcal{M}$ has two disjoint bases B_1 and B_2. Note that fixing an element is like contracting an element in the matroid and removing one element is like deleting the element from the matroid. From Lecture 9, we know that for any two subsets E and F of $\mathcal{M}$, $(\mathcal{M} \setminus E)/F = (\mathcal{M}/F) \setminus E$. This means that the order of deleting or contracting (fixing) the elements does not matter. After k moves of both players, let $E = \{e_1, e_2, \ldots, e_k\}$ be the set of elements that player 1 has deleted and $F = \{f_1, f_2, \ldots, f_k\}$ be the set of elements that player 2 has fixed. We prove by induction that player 2 can play in such a way that after his move, there still exist two disjoint bases A_1 and A_2 in the remaining matroid $\mathcal{M}' = (\mathcal{M} \setminus E)/F$. By assumption, the base of the induction is true by taking $\mathcal{M}' = \mathcal{M}$ and $A_1 = B_1$ and $A_2 = B_2$. We assume there exist two disjoint bases A_1 and A_2 in $\mathcal{M}' = (\mathcal{M} \setminus E)/F$. Now, if player 1 deletes element e_{k+1}, say from A_1, then from the basis exchange property, we can find $f_{k+1} \in A_2$ such that $A_1 - \{e_{k+1}\} + f_{k+1} \in \mathcal{I}'$. Restated, this means that $A_1 - \{e_{k+1}\}$ is a basis for $\mathcal{M}' \setminus \{e_{k+1}\}/\{f_{k+1}\}$, and so is $A_2 - \{f_{k+1}\}$. We therefore have two disjoint bases in $\mathcal{M}' \setminus \{e_{k+1}\}/\{f_{k+1}\}$, and we can proceed.

$\square$