Chapter 8

Gauss Map I

8.1 "Curvature" of a Surface

We've already discussed the curvature of a curve. We'd like to come up with an analogous concept for the "curvature" of a regular parametrized surface S parametrized by $x\colon U \to \mathbb{R}^n$. This can't be just a number — we need at the very least to talk about the "curvature of S at p in the direction $v \in T_p(S)$".

So given $v \in T_p(S)$, we can take a curve $\alpha\colon I \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = v$. (This exists by the definition of the tangent plane.) The curvature of α itself as a curve in $\mathbb{R}^n$ is $\frac{d^2\alpha}{ds^2}$ (note that this is with respect to arc length). However, this depends on the choice of α — for example, if you have the cylinder over the unit circle, and let v be in the tangential direction, both a curve that just goes around the cylinder and a curve that looks more like a parabola that happens to be going purely tangentially at p have the same α', but they do not have the same curvature. But if we choose a field of normal vectors N on the surface, then $\frac{d^2\alpha}{ds^2} \cdot N_p$ is independent of the choice of α (as long as $\alpha(0) = p$ and $\alpha'(0) = v$). It's even independent of the magnitude of v — it only depends on its direction $\hat{v}$. We call this curvature $k_p(N, \hat{v})$. For the example, we can see that the first curve's α'' is 0, and that the second one's α'' points in the negative $\hat{z}$ direction, whereas N points in the radial direction, so $k_p(N, \hat{v})$ is zero no matter which α you choose.

(In 3-space with a parametrized surface, we can always choose N to be $N = \frac{x_u \wedge x_v}{|x_u \wedge x_v|}$.)

To prove this, we see that $\alpha(s) = x(u_1(s), u_2(s))$, so that $\frac{d\alpha}{ds} = \frac{du_1}{ds} x_{u_1} + \frac{du_2}{ds} x_{u_2}$ and $\frac{d^2\alpha}{ds^2} = \frac{d^2u_1}{ds} x_{u_1} + \frac{du_1}{ds}\left(\frac{du_1}{ds} x_{u_1u_1} + \frac{du_2}{s} x_{u_1u_2}\right) + \frac{d^2u_2}{ds} x_{u_2} + \frac{du_2}{ds}\left(\frac{du_1}{ds} x_{u_1u_2} + \frac{du_2}{s} x_{u_2u_2}\right)$. But by normality, $N \cdot x_{u_1} = N \cdot x_{u_2} = 0$, so $\frac{d^2\alpha}{ds^2} \cdot N = \sum_{i,j=1}^{2} b_{ij}(N) \frac{du_i}{ds}\frac{du_j}{ds}$, where $b_{ij}(N) = x_{u_iu_j} \cdot N$.

We can put the values b_{ij} into a matrix $B(N) = [b_{ij}(N)]$. It is symmetric, and so it defines a symmetric quadratic form $B = II\colon T_p(S) \to \mathbb{R}$. If we use $\{x_{u_1}, x_{u_2}\}$ as a basis for $T_p(S)$, then $II(cx_{u_1} + dx_{u_2}) = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} b_{11}(N) & b_{12}(N) \\ b_{21}(N) & b_{22}(N) \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$. We call II the *Second Fundamental Form.*

II is independent of α, since it depends only on the surface (not on α). To show that $k_p(N, \hat{v})$ is independent of choice of α, we see that

$$k_p(N, \hat{V}) = \frac{d^2\alpha}{ds^2} \cdot N = \sum_{ij} b_{ij}(N) \frac{du_i}{ds}\frac{du_j}{ds} = \frac{\sum_{i,j} b_{ij}(N) \frac{du_i}{dt}\frac{du_j}{dt}}{\left(\frac{ds}{dt}\right)^2}$$

Now, $s(t) = \int_{t_0}^{t} |\alpha'(t)|\, dt$, so that $\left(\frac{ds}{dt}\right)^2 = |\alpha'(t)|^2 = |\frac{du_1}{dt} x_{u_1} + \frac{du_2}{dt} x_{u_2}|^2 = \sum_{i,j} \left(\frac{du_i}{dt}\right)\left(\frac{du_j}{dt}\right) g_{ij}$, where g_{ij} comes from the *First* Fundamental Form. So

$$k_p(N, \hat{v}) = \frac{\sum_{i,j} b_{ij}(N) \frac{du_i}{dt}\frac{du_j}{dt}}{\sum_{i,j} g_{ij} \frac{du_i}{dt}\frac{du_j}{dt}}$$

The numerator is just the First Fundamental Form of v, which is to say its length. So the only property of α that this depends on are the derivatives of its components at p, which are just the components of the given vector v. And in fact if we multiply v by a scalar λ, we multiply both the numerator and the denominator by λ^2, so that $k_p(N, \hat{v})$ doesn't change. So $k_p(N, \hat{v})$ depends only on the *direction* of v, not its magnitude.

If we now let $k_1(N)_p$ be the maximum value of $k_p(N, \hat{v})$. This exists because $\hat{v}$ is chosen from the compact set $S^1 \subset T_p(S)$. Similarly, we let

$k_2(N)_p$ be the minimal value of $k_p(N, \hat{v})$. These are called the *principle curvatures* of S at p with regards to N. The directions e_1 and e_2 yielding these curvatures are called the *principal directions*. It will turn out that these are the eigenvectors and eigenvalues of a linear operator defined by the Gauss map.

8.2 Gauss Map

Recall that for a surface $x\colon U \to S$ in $\mathbb{R}^3$, we can define the Gauss map $N\colon S \to S^2$ which sends p to $N_p = \frac{x_{u_1} \wedge x_{u_2}}{|x_{u_1} \wedge x_{u_2}|}$, the unit normal vector at p. Then $dN_p\colon T_p(S) \to T_{N_p}(S^2)$; but $T_p(S)$ and $T_{N_p}(S^2)$ are the same plane (they have the same normal vector), so we can see this as a linear operator $T_p(S) \to T_p(S)$.

For example, if $S = S^2$, then $N_p = p$, so N_p is a linear transform so it is its own derivative, so dN_p is also the identity.

For example, if S is a plane, then N_p is constant, so its derivative is the zero map.

For example, if S is a right cylinder defined by $(\theta, z) \mapsto (\cos\theta, \sin\theta, z)$, then $N(x, y, z) = (x, y, 0)$. (We can see this because the cylinder is defined by $x^2 + y^2 = 1$, so $2xx' + 2yy' = 0$, which means that $(x, y, 0) \cdot (x', y', z') = 0$, so that $(x, y, 0)$ is normal to the velocity of *any* vector through (x, y, z).) Let us consider the curve α with $\alpha(t) = (\cos t, \sin t, z(t))$, then $\alpha'(t) = (-\sin t, \cos t, z'(t))$. So $(N \circ \alpha)(t) = (x(t), y(t), 0)$, and so $(N \circ \alpha)'(t) = (-\sin t, \cos t, 0)$. So $dN_p(x_\theta) = x_\theta$. So in the basis $\{x_\theta, x_z\}$, the matrix is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. It has determinant 0 and $\frac{1}{2}$ trace equal to $\frac{1}{2}$. It turns out that the determinant of this matrix only depends on the First Fundamental Form, and not how it sits in space — this is why the determinant is the same for the cylinder and the plane. A zero eigenvalue can't go away no matter how you curve the surface, as long as you don't stretch it.