Chapter 10

Introduction to Minimal Surfaces I

10.1 Calculating the Gauss Map using Coordinates

Last time, we used the differential of the Gauss map to define several interesting features of a surface — mean curvature H, Gauss curvature K, and principal curvatures k_1 and k_2. We did this using relatively general statements. Now we will calculate these quantities in terms of the entries g_{ij} and b_{ij} of the two fundamental form matrices. (Note that do Carmo still uses E, F, G, and e, f, g here respectively.) Don't forget that the terms g_{ij} and $b_{ij}(N)$ can be calculated with just a bunch of partial derivatives, dot products, and a wedge product — the algebra might be messy but there's no creativity required.

Let $dN_p = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in terms of the basis $\{\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\}$ of $T_p(S)$. Now, $\frac{\partial N}{\partial u} = a_{11}\frac{\partial x}{\partial u} + a_{21}\frac{\partial x}{\partial v}$; so $\left\langle \frac{\partial N}{\partial u}, \frac{\partial x}{\partial u}, = \right\rangle a_{11}g_{11} + a_{21}g_{12}$. But by a proof from last time, $\left\langle \frac{\partial N}{\partial u}, \frac{\partial x}{\partial u}, = \right\rangle - \left\langle N, \frac{\partial^2 x}{\partial u^2}, = \right\rangle - b_{11}(N)$. So $-b_{11}(N) = a_{11}g_{11} + a_{21}g_{12}$.

Three more similar calculations will show us that

$$-\big[b_{ij}(N)\big] = \big[a_{ij}\big]\big[g_{ij}\big]$$

If we recall that the Gaussian curvature $K = k_1 k_2$ is the determinant of $dN_p = \big(a_{ij}\big)$, then we can see that $\det\big[b_{ij}(N)\big] = K \det\big[g_{ij}\big]^{-1}$, so that $K = \frac{b_{11}(N)b_{22}(N) - b_{12}(N)^2}{g_{11}g_{22} - g_{12}^2}$.

If we solve the matrix equality for the matrix of a_{ij}, we get that

$$\big[a_{ij}\big] = \frac{1}{\det G}\begin{bmatrix} g_{12}b_{12}(N) - g_{22}b_{11}(N) & g_{12}b_{22}(N) - g_{22}b_{12}(N) \\ g_{12}b_{11}(N) - g_{11}b_{12}(N) & g_{12}b_{12}(N) - g_{11}b_{22}(N) \end{bmatrix}$$

We recall that $-k_1$ and $-k_2$ are the eigenvalues of dN_p. Thus, for some nonzero vector v_i, we have that $dN_p(v_i) = -k_i v_i = -k_i I v_i$. Thus $\begin{bmatrix} a_{11} + k_i & a_{12} \\ a_{21} & a_{22} + k_i \end{bmatrix}$ maps some nonzero vector to zero, so its determinant must be zero. That is, $k_i^2 + k_i(a_{11} + a_{22}) + a_{11}a_{22} - a_{21}a_{12} = 0$; both k_1 and k_2 are roots of this polynomial. Now, for any quadratic, the coefficient of the linear term is the opposite of the sum of the roots. So $H = \frac{1}{2}(k_1 + k_2) = -\frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2}\frac{b_{11}(N)g_{22} - 2b_{12}(N)g_{12} + b_{22}(N)g_{11}}{g_{11}g_{22} - g_{12}^2}$. (This is going to be the Really Important Equation.)

Last, we find the actual values k_1 and k_2. Remembering that the constant term of a quadratic is the product of its roots and thus K, which we've already calculated, we see that the quadratic we have is just $k_i^2 - 2Hk_i + K = 0$; this has solutions $k_i = H \pm \sqrt{H^2 - K}$.

As an exercise, calculate the mean curvature H of the helicoid $x(uv) = (v\cos u,\ v\sin u,\ cu)$. (This was in fact a homework problem for today, but work it out again anyway.)

10.2 Minimal Surfaces

Since the $b_{ij}(N)$ are defined as the dot product of N and something independent of N, they are each linear in N. So $H(N) = \frac{1}{2}\frac{b_{11}(N)g_{22}-2b_{12}(N)g_{12}+b_{22}(N)g_{11}}{g_{11}g_{22}-g_{12}^2}$ is also linear in N. We can actually consider mean curvature as a vector H instead of as a function from N to a scalar, by finding the unique vector H such that $H(N) = H \cdot N$. I'm pretty sure that this is more interesting when we're embedded in something higher than $\mathbb{R}^3$.

We define a surface where $H = 0$ everywhere to be a *minimal surface*. Michael Nagle will explain this choice of name next time. You just calculated that the helicoid is a minimal surface. So a surface is minimal iff $g_{22}b_{11}(N) + g_{11}b_{22}(N) - 2g_{12}b_{12}(N) = 0$.

Another example of a minimal surface is the catenoid: $x(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$. (We've looked at this in at least one homework exercise.) We calculate $\frac{\partial x}{\partial u} = (-\cosh v \sin u, \cosh v \cos u, 0)$ and $\frac{\partial x}{\partial v} = (\sinh v \cos u, \sinh v \sin u, 1)$, so that $\left[g_{ij}\right] = \begin{bmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{bmatrix}$. Next, $\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v} = (\cosh v \cos u, \cos v \sin u, -\cosh v \sinh v)$, with norm $\cosh^2 v$. So $N_p = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, -\tanh v\right)$.

Taking the second partials, $\frac{\partial^2 x}{\partial u^2} = (-\cosh v \cos u, -\cosh v \sin u, 0)$, $\frac{\partial^2 x}{\partial v^2} = (\cosh v \cos u, \cosh v \sin u, 0)$, and $\frac{\partial^2 x}{\partial u \partial v} = (-\sinh v \sin u, \sinh v \cos u, 0)$. So $\left[b_{ij}(N)\right] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Finally, the numerator of H is $g_{22}b_{11}(N) + g_{11}b_{22}(N) - 2g_{12}b_{12}(N) = -\cosh^2 v + \cosh^2 v - 0 = 0$. So the catenoid is a minimal surface. In fact, it's the *only* surface of rotation that's a minimal surface. (Note: there are formulas in do Carmo for the second fundamental form of a surface of rotation on page 161, but they assume that the rotating curve is parametrized by arc length, so they'll give the wrong answers for this particular question.)

Why is it the only one? Say we have a curve $y(x) = f(x)$ in the xy-plane Let S be the surface of rotation around the x-axis from this. We can show that the lines of curvature of the surface are the circles in the yz-plane and the lines of fixed θ. We can show that the first have curvature $\frac{1}{y}\frac{1}{(1+(y')^2)^{\frac{1}{2}}}$, and the second have the same curvature as the graph y, which is $\frac{y''}{(1+(y')^2)^{\frac{3}{2}}}$. So H is the sum of these: $\frac{1+(y')^2-yy''}{2y(1+(y')^2)^{\frac{3}{2}}}$. So this is 0 if $1+\left(\frac{dy}{x}\right)^2 - y\frac{d^2y}{dx^2} = 0$. If we let $p = \frac{dy}{dx}$, then $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$. So our equation becomes $1+p^2 - yp\frac{dp}{dy} = 0$, or $\frac{p}{1+p^2}dp = \frac{1}{y}dy$. Integrating, we get $\frac{1}{2}\log(1+p^2) = \log y + C$, so that $y = C_0\sqrt{1+p^2}$. Then $p = \frac{dy}{dx} = \sqrt{cy^2-1}$, so that $\frac{dy}{\sqrt{cy^2-1}} = dx$. Integrating (if you knew this!), you get $\frac{\cosh^{-1} cy}{c} = x+k$, which is to say that $y = c\cosh\frac{x+l}{c}$. Whew!