SYMPLECTIC GEOMETRY, LECTURE 25

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1. Spin Structures

Let (X^4, g) be an oriented Riemannian manifold, $S = S_+ \oplus S_- \to X$ a spin^c structure with Clifford multiplication $\gamma : T^*X \otimes S \to S$.

Example. If X is almost-complex, $S_+ = (\bigwedge^{0,0} \otimes E) \oplus (\bigwedge^{0,2} \otimes E)$, $S_- = (\bigwedge^{0,1} \otimes E)$, $\gamma(u) = \sqrt{2}[u^{0,1} \wedge \cdot - \iota_{(u^{1,0})^\#} \cdot]$. As defined last time, $L = \det\,(S_+) = \det\,(S_-) = K_X^{-1} \otimes E^2$.

As we stated last time, the Clifford multiplication extends to differential forms with $\bigwedge^2_+ \cong \mathrm{End}_{TLAH}(S^+)$ (where the latter group is the space of traceless, anti-hermitian endomorphisms). We also have the Dirac operator associated to a spin^c connection ∇^A on S:

$$D_A : \Gamma(S^{\pm}) \to \Gamma(S^{\mp}), D_A\psi = \sum_i \gamma(e^i)(\nabla^A_{e_i}\psi) \tag{1}$$

Example. If X is Kähler, the spin^c connection is induced by ∇_a connection on E, and $D_A = \sqrt{2}(\overline{\partial}_a + \overline{\partial}_a^*)$.

Example. $\nabla^A = \nabla^{A_0} + ia \otimes \mathrm{id}$ on $S_\pm$ for $a \in \Omega^1$ corresponding to $A = A_0 + 2ia$ on L. The associated decomposition of the Dirac operator is $D_A = D_{A_0} + \gamma(a)$.

2. Seiberg-Witten Equations

Definition 1. *The* Seiberg-Witten equations *are the equations*

$$\begin{aligned} D_A\psi &= 0 \in \Gamma(S^-) \\ \gamma(F_A^+) &= (\psi^* \otimes \psi)_0[+\gamma(\mu)] \in \Gamma(\mathrm{End}(S^+)) \end{aligned} \tag{2}$$

where A is a Hermitian connection on $L = \bigwedge^2 S^{\pm}$ (corresponding to a spin^c *connection ∇^A), $\psi \in \Gamma(S+)$ is a section, $F_A^+ = \frac{1}{2}(F_A + *F_A) \in i\Omega^2_+$ for $F_A \in i\Omega^2$ the curvature of A, $(\psi^* \otimes \psi)_0$ is the traceless part of $\psi^* \otimes \psi$, and μ is an imaginary self-dual form fixed in advance.*

Now, there exists an ∞-dimensional group of symmetries preserving solutions, called the *gauge group* $\mathcal{G} = C^\infty(X, S^1)$ where $f \in C^\infty(X, S^1)$ acts by

$$(A, \psi) \mapsto (A - 2df \cdot f^{-1}, f\psi) \tag{3}$$

Proposition 1. *This preserves the solution space, and the action of $\mathcal{G}$ is free unless $\psi \equiv 0$ (*reducible solutions*), where* $\mathrm{Stab}((A, 0)) \cong S^1$ *is the space of constant maps.*

Reducible solutions can happen $\Leftrightarrow F_A^+ = \mu$ has a solution $\Leftrightarrow (g, \mu)$ lie in a codimension b_2^+ subspace. Thus, we want to assume $b_2^+(X) \geq 1$, and (g, μ) generic. Note that, for $\mu = 0, F_A^+ = 0 \Leftrightarrow \frac{i}{2\pi}F_A$ is closed and antiselfdual in the class $c_1(L) \in \mathcal{H}^2_- \subset \mathcal{H}^2_- \oplus \mathcal{H}^2_+ = H^2$.

Definition 2. *The* moduli space of solutions $\mathcal{M}(S, g, \mu)$ *is the set of solutions modulo $\mathcal{G}$.*

Theorem 1. *For (g, μ) generic, $\mathcal{M}$ (if nonempty) is a smooth, compact, orientable manifold of dimension*

$$d(S) = \frac{1}{4}(c_1(L)^2 \cdot [X] - (2\chi + 3\sigma)) \tag{4}$$

Idea: We want to understand, given a solution (A_0, ψ_0) to the SW equations, the nearby solutions to the same equations. We linearize the SW equations, and let $(a, \phi) \in \Omega^1(X, i\mathbb{R}) \times C^\infty(S^+)$ be a small change in the solution, obtaining

(5) $$P_1 : (a, \phi) \mapsto D_{A_0}\phi + \gamma(a)\psi_0$$

as the linearization of the first equation and

(6) $$P_2 : (a, \phi) \mapsto \gamma((da)^+) - (\phi \otimes \psi_0^* + \psi_0 \otimes \phi^*)_0$$

as the linearization of the second equation. We restrict $P = P_1 \oplus P_2$ to a slice transverse to the $\mathcal{G}$-action $A \mapsto A - 2df \cdot f^{-1}, \psi \mapsto f\psi$, i.e. to $\mathcal{S} = \{(a, \phi) | d^*a = 0 \text{ and } \mathrm{Im}(\langle \phi, \psi_0 \rangle_{L^2}) = 0\}$ (which is transverse to the $\mathcal{G}$-orbit at (A_0, ψ_0)). Then $P|_{\text{Ker } d^* \times L^2_1(S^+)}$ is a differential operator of order 1, and is Fredholm (f.d. kernel and cokernel) since

(7) $$(P \oplus d^*) : L^2_2(X, i\wedge^1) \times L^2_1(S^+) \to L^2(S^-) \times L^2_1(X, i\wedge^2_+) \times L^2_1(X, i\mathbb{R})$$

$(= D_{A_0} \oplus (d^+ \oplus d^*) + \text{order } 0)$ is elliptic. Elliptic regularity implies that both Ker , Coker lie in C^∞. For generic (g, μ), P is surjective (specifically, consider $\{(A, \psi, \mu) | \cdots\}/\mathcal{G}$ and apply Sard's theorem to project to μ and find a good choice). We expect that Ker P is the tangent space to $\mathcal{M}$: this is only ok if Coker $P = 0$, so we can use the implicit function theorem to show that $\mathcal{M}$ is smooth with $T\mathcal{M} = \text{Ker } P|_{\mathcal{S}}$. The statement about the dimension follows from the *Atiyah-Singer index theorem*, which gives a formula for $d(S) = \mathrm{ind}(P|_{\mathcal{S}}) =$ dim Ker $-$ dim Coker . Compactness of $\mathcal{M}$ follows from the a priori bounds on the solutions: the key point is that we get a bound on $\sup |\psi|$, so elliptic regularity and "bootstrapping" give us bounds in all norms.

Consider a solution (A, ψ) of the SW equations (for simplicity assume $\mu = 0$). We have the following Weitzenbock formula for the Dirac operator:

(8) $$D_A^2\psi = \nabla_A^*\nabla_A\psi + \frac{s}{4}\psi + \frac{1}{2}\gamma(F_A^+)\psi$$

where ∇_A^* is the L^2-adjoint of ∇_A, s is the scalar curvature of the metric g (this can be shown by calculation in a local frame). Now,

(9) $$D_A\psi = 0 \implies 0 = \langle D_A^2\psi, \psi\rangle = \langle \nabla_A^*\nabla_A\psi, \psi\rangle + \frac{s}{4}|\psi|^2 + \frac{1}{2}\langle\gamma(F_A^+)\psi, \psi\rangle$$

where $\gamma(F_A^+) = (\psi^* \otimes \psi)_0 = \psi^* \otimes \psi - \frac{1}{2}|\psi|^2$. Then

(10) $$0 = \frac{1}{2}d^*d|\psi|^2 + |\nabla_A\psi|^2 + \frac{s}{4}|\psi|^2 + \frac{1}{4}|\psi|^4$$

Take a point where $|\psi|$ is maximal. Then

(11) $$\frac{1}{2}d^*d|\psi|^2 \geq 0 \implies \frac{s}{4}|\psi|^2 + \frac{1}{4}|\psi|^4 \leq 0 \implies |\psi|^2 \leq \max(-s, 0)$$

Theorem 2. *If g has scalar curvature > 0, then the SW-invariants $\equiv 0$.*

Proof. A small generic perturbation ensures that there are no reducible solutions. The above estimate on $\sup|\psi|$ ensures that there are no irreducible solutions either. □