SYMPLECTIC GEOMETRY, LECTURE 14

Prof. Denis Auroux

1. Kähler Geometry

Let (M, ω, J) be a Kähler manifold, with ω a symplectic form and J an integrable complex structure compatible with ω.

- Compatibility $\omega(Ju, Jv) = \omega(u, v)$: note that, for a $(2,0)$-form $\gamma = \sum a_{i,j} dz_i \wedge dz_j$, we have $\gamma(Ju, Jv) = -\gamma(u, v)$, and similarly for a $(0,2)$-form. For a $(1,1)$-form $\gamma = \sum a_{i,j} dz_i \wedge d\overline{z}_j$, we have $\gamma(Ju, Jv) = \gamma(u, v)$, implying that $\omega \in \Omega^{1,1}$.
- Closedness $d\omega = 0 \Leftrightarrow \partial\omega = 0, \overline{\partial}\omega = 0$: in particular, $[\omega] \in H^{1,1}_{\overline{\partial}}(M)$ lives in the Dolbeault cohomology of M. Moreover ω is real (i.e. $\overline{\omega} = \omega$). Writing ω locally as $\frac{i}{2}\sum_{j,k=1}^n h_{jk} dz_j \wedge d\overline{z}_k$, so

$$\overline{\omega} = \frac{i}{2}\sum_{j,k=1}^{n} \overline{h_{jk}} dz_k \wedge d\overline{z}_j \tag{1}$$

  we have that $h_{jk} = \overline{h_{kj}}$, and (h_{jk}) must be a Hermitian matrix.
- Nondegeneracy $\omega^n \neq 0 \Leftrightarrow (h_{jk})$ is invertible, since

$$\omega^n = \pm(\frac{i}{2})^n n!(\det\ (h_{jk}))dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n \tag{2}$$

- Positivity $\omega(v, Jv) > 0 \Leftrightarrow$ positivity of $g(\cdot,\cdot) = \omega(\cdot, J\cdot) = \sum h_{jk} dz_j d\overline{z}_k \Leftrightarrow (h_{jk})$ is a positive definite Hermitian matrix.

Thus, we find that, given a complex manifold (M, J), ω is a Kähler form $\Leftrightarrow \omega \in \Omega^{1,1}_{\mathbb{R}}, \overline{\partial}\omega = 0$, and locally $\omega = \frac{1}{2}\sum h_{jk} dz_j \wedge d\overline{z}_k$ for (h_{jk}) a positive definite Hermitian matrix. Moreover, since these properties are preserved by convex linear combinations, any two Kähler forms for the same complex structure J are deformation equivalent and isotopic if $[\omega]$ is fixed.

1.1. Kähler potential.

Definition 1. *For M a complex manifold, $\phi \in C^\infty(M, \mathbb{R})$ is strictly plurisubharmonic (spsh) if on each complex chart (U, z_j), the matrix $(\frac{\partial\phi}{\partial z_j \partial \overline{z}_k})$ is positive definite at every point.*

Recall that J integrable, $d^2 = 0 \implies \partial^2 = 0, \partial\overline{\partial} + \overline{\partial}\partial = 0, \overline{\partial}^2 = 0$.

Proposition 1. *ϕ spsh $\Leftrightarrow \frac{i}{2}\partial\overline{\partial}\phi$ is Kähler.*

Example. On $\mathbb{C}^n$, $\phi = \sum |z_j|^2 = \sum z_j \overline{z_j}$ is strictly plurisubharmonic since $(\frac{\partial\phi}{\partial z_j \partial \overline{z}_k})$ is the identity matrix, and the corresponding symplectic form $\omega = \frac{i}{2}\sum dz_j \wedge d\overline{z}_j$ is the standard one.

We have the following converse.

Theorem 1. *For ω a closed, real-valued $(1,1)$-form on $p \in M$, $\exists$ a neighborhood $U \ni p$, $\phi \in C^\infty(U, \mathbb{R})$ s.t. $\omega = \frac{i}{2}\partial\overline{\partial}\phi$. This ϕ is called a* local Kähler potential *for ω.*

1.2. **Examples of Kähler Manifolds.**

Example. Any complex submanifold of $(\mathbb{C}^n, \omega)$ is Kähler, with the inherited complex and symplectic structures.

Example. Complex projective space $\mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*$ is Kähler: letting

$$U_i = \{(z_0 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n)\} \tag{3}$$

be the standard charts $\cong \mathbb{C}^n$, we have the *Fubini-Study* Kähler form

$$\omega_{FS} = \frac{i}{2}\partial\overline{\partial}\log(1+|z|^2) \tag{4}$$

(since $f(z) = \log(1+|z|)^2$ is spsh). Explicitly,

$$\partial\overline{\partial}f = \partial\frac{\sum z_j d\overline{z}_j}{1+|z|^2} = \frac{(1+|z|^2)\sum dz_j \wedge d\overline{z}_j - (\sum \overline{z}_j dz_j)\wedge(\sum z_j d\overline{z}_j)}{(1+|z|^2)^2} \tag{5}$$

Applying this to $v \in T^{1,0}, \overline{v} \in T^{0,1}$, we obtain

$$\frac{(1+|z|^2)\,|v|^2 - |\langle z, v\rangle|^2}{(1+|z|^2)^2} \geq \frac{|v|^2}{(1+|z|^2)^2} \tag{6}$$

Since $\frac{i}{2}\partial\overline{\partial}f(u, iu) = \partial\overline{\partial}f(u^{1,0}, \overline{u^{1,0}})$, we have the desired positivity. Moreover, for ϕ a transition map (WLOG between U_0 and U_1), we have that $\phi^* f = \log(1+|z|^2) - \log|z_1|^2 \implies \partial\overline{\partial}(\phi^* f) = \partial\overline{\partial}f$ since

$$\partial\overline{\partial}\log|z_1|^2 = \partial\frac{z_1 d\overline{z}_1}{|z_1|^2} = \partial\frac{d\overline{z_1}}{\overline{z_1}} = 0 \tag{7}$$

Finally, recall that $H^2(\mathbb{CP}^n, \mathbb{R}) = \mathbb{R}$, and $H_2(\mathbb{CP}^n)$ is generated by $[\mathbb{CP}^1]$. The class of $[\omega]$ is thus defined by the value of

$$[\omega]\cdot[\mathbb{CP}^1] = \int_{\mathbb{CP}^1}\omega_{FS} = \mathrm{Area}(\mathbb{CP}^1, \omega_{FS}) \tag{8}$$

Example. Any complex submanifold of $\mathbb{CP}^n$ (i.e. complex projective variety) is Kähler.

Theorem 2 (Kodaira Embedding)**.** *Let (X, ω, J) be a compact Kähler manifold, with $[\omega] \in H^2(X, \mathbb{R})$ an integral class. Then $\exists$ a holomorphic embedding $X \hookrightarrow \mathbb{CP}^n$ making it a complex projective variety, with ω differing from ω_{FS} by a scaling factor.*

Theorem 3 (Hodge)**.** *For (M, ω) a compact Kähler manifold, the Dolbeault cohomology groups $H^{p,q}_{\overline{\partial}}(M)$ satisfy $H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M)$ and $H^{p,q} \cong \overline{H^{q,p}}$.*

Corollary 1. dim $H^k(M)$ *is even for odd k.*

Example. In the 70's, Kodaira and Thurston independently studied a closed 4-manifold which carries both a complex structure and a symplectic structure but which is not Kähler.