SYMPLECTIC GEOMETRY, LECTURE 3

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1. Symplectic Manifolds

Let (M,ω) be a symplectic manifold, i.e. a smooth manifold with nondegenerate closed 2-form ω.

Example. For X a smooth manifold, the cotangent bundle $M = T^*X$ is a symplectic manifold. Specifically, given a chart $U \subset X$ with coordinates $x_1, \dots, x_n$, we have a basis of T_p^*X given by $dx_1, \dots, dx_n$ and every $\xi \in T^*X$ can be written as $\sum \xi_i dx_i$. This gives us a map

(1) $$T^*X|_U \to \mathbb{R}^{2n}, (x,\xi) \mapsto (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$$

Let α be the Liouville form defined by $\sum \xi_i dx_i$ on each coordinate patch. It is well-defined as a 1-form on M, and $\omega = d\alpha = \sum d\xi_i \wedge dx_i$ is the desired symplectic form. Furthermore, given a diffeomorphism $X_1 \to X_2$, we have an induced map

(2) $$F : T^*X_1 \to T^*X_2, (x,\xi) \mapsto (f(x), (d_x f)^{-1*}\xi)$$

which is a symplectomorphism (because $\exists$ local coordinates in which f is the identity). Also, given $h \in C^\infty(X,\mathbb{R})$, we have an associated symplectomorphism $\tau_h : M \to M, (x,\xi) \mapsto (x, \xi + d_x h)$ since

(3) $$\tau_h^*\alpha = \alpha + dh \implies \tau_h^*\omega = \tau_h^*(d\alpha) = d\alpha + ddh = \omega$$

as desired.

1.1. Submanifolds.

Definition 1. *A submanifold $W \subset (M,\omega)$ is* symplectic *if $\omega|_W$ is symplectic (specifically, nondegenerate). This implies that $T_pW \subset T_pM$ is a symplectic subspace $\forall p$. $L \subset (M,\omega)$ is* Lagrangian *if $\omega|_L = 0$ and $\dim L = \frac{1}{2}\dim M$.*

Example. By our above construction, the 0-section $X \hookrightarrow T^*X = M$ is a Lagrangian submanifold. Furthermore, sections of T^*X are graphs $X_\mu = \{(x,\mu(x)) | x \in X\} \subset T^*X$ of 1-forms $\mu \in \Omega^1(X,\mathbb{R})$: such a graph is Lagrangian iff $d\mu = 0$, since denoting $i_\mu(x) = (x,\mu(x))$, $i_\mu^*\alpha = \mu \implies i_\mu^*(\omega) = i_\mu^*(d\alpha) = di_\mu^*\alpha = d\mu$.

Example. For $\Sigma^k \subset X^n$ a submanifold, define the conormal space to $x \in \Sigma$ by

(4) $$N_x^*\Sigma = \{\xi \in T_x^*X | \xi|_{T_x\Sigma} = 0\}$$

This gives us subbundle $N^*\Sigma \subset T^*X|_\Sigma$ and a submanifold $N^*\Sigma \subset T^*X$. For $\Sigma = X$, we get the 0-section: for $\Sigma = \{p\}$, we get the fiber T_p^*X. By definition, $\alpha|_{N^*\Sigma} = 0$, so $N^*\Sigma$ is Lagrangian.

1.2. Symplectomorphisms and Lagrangian Submanifolds.

Let $\phi : (M_1,\omega_1) \to (M_2,\omega_2)$ be a diffeomorphism: we want to know whether ϕ is a symplectomorphism as well, i.e. whether $\phi^*\omega_2 = \omega_1$. Consider the graph $\Gamma_\phi \subset M = M_1 \times M_2$. The latter space has one symplectic structure via $\omega = \omega_1 \oplus \omega_2 = \pi_1^*\omega_1 + \pi_2^*\omega_2$, which is nondegenerate since

(5) $$\omega^{n_1+n_2} = \binom{n_1+n_2}{n_1}\pi_1^*\omega_1^{n_1} \wedge \pi_2^*\omega_2^{n_2}$$

However, here we will consider the alternate symplectic structure given by $\hat\omega = \pi_1^*\omega_1 - \pi_2^*\omega_2$.

Proposition 1. *ϕ is a symplectomorphism $\Leftrightarrow \Gamma_\phi$ is Lagrangian.*

Proof. Γ_ϕ is the image of the embedding $\gamma : M_1 \to M_1 \times M_2, p \mapsto (p,\phi(p))$, and $\gamma^*\hat\omega = \gamma^*\pi_1^*\omega_1 - \gamma^*\pi_2^*\omega_2 = \omega_1 - \phi^*\omega_2$ is 0 $\Leftrightarrow \Gamma_\phi$ is Lagrangian. ◻

2. Hamiltonian Vector Fields

Let M be a manifold.

Definition 2. *An* isotopy *on M is a C^∞ map $\rho : M \times \mathbb{R} \to M$ s.t. $\rho_0 = \mathrm{id}$ and $\forall t, \rho_t$ is a diffeomorphism.*

Given an isotopy, we obtain a time-dependent vector field $v_t : p \mapsto \frac{d}{ds}\rho_s(q)|_{s=t}$ where $q = \rho_t^{-1}(p)$. We say that ρ_t is the *flow* of v_t. Conversely, if M is compact or v_t is sufficiently "good", we can integrate to obtain the flow from the vector field. If v is time-independent, we obtain a 1-parameter group $\rho_t = \exp(tv)$, with associated vector field v. Recall the Lie derivative $L_v\alpha = \frac{d}{dt}(\exp(tv)^*\alpha)|_{t=0}$.

Proposition 2 (Cartan's Formula)**.** *$L_v\alpha = di_v\alpha + i_v d\alpha$.*

If (ρ_t) is generated by (v_t) then $\frac{d}{dt}(\rho_t^*\alpha) = \rho_t^*(L_{v_t}\alpha)$.

Now, let (M, ω) be a symplectic manifold, $H : M \to \mathbb{R}$ a C^∞ map. Then $dH \in \Omega^1(M) \implies \exists$ a unique vector field X_H s.t. $i_{X_H}\omega = dH$, called the *Hamiltonian vector field* generated by H (H itself is called the *Hamiltonian function*). Now, assume that M is compact, or that the flow of X_H is well-defined. Then we obtain an isotopy $\rho_t : M \to M$ of diffeomorphisms generated by X_H.

Proposition 3. *ρ_t are symplectomorphisms.*

Proof. Note that $\frac{d}{dt}(\rho_t^*\omega) = \rho_t^*(L_{X_H}\omega)$ but $L_{X_H}\omega = di_{X_H}\omega + i_{X_H}d\omega = d^2H = 0$. Since ρ_0 is the identity, $\rho_t^*\omega = \omega$ for all t. □

Example. For $\mathbb{R}^{2n}$ with coordinates $x_1, \dots, x_n, p_1, \dots, p_n$, the function $H(x,p) = \frac{1}{2}|p|^2 + V(x)$ has derivative $dH = \sum p_i dp_i + \frac{\partial V}{\partial x_i}dx_i$. Thus, the associated vector field is $X_H = \sum -p_i \frac{\partial}{\partial x_i} + \frac{\partial V}{\partial x_i}\frac{\partial}{\partial p_i}$, giving us *Hamilton's equations*

$$\frac{dx_i}{dt} = -p_i = -\frac{\partial H}{\partial p_i}, \; \frac{dp_i}{dt} = \frac{\partial V}{\partial x_i} = \frac{\partial H}{\partial x_i} \tag{6}$$