18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Steenrod Operations (Lecture 2)

The objective of today's lecture is to introduce the Steenrod operations and establish some of their basic properties. We will work over the finite field $\mathbf{F}_2 \simeq \mathbf{Z}/2\mathbf{Z}$ with two elements.

To this end, we will study the homotopy theory of cochain complexes

$$\ldots \rightarrow V^{n-1} \stackrel{d_{n-1}}{\rightarrow} V^n \stackrel{d_n}{\rightarrow} V^{n+1} \rightarrow \ldots$$

in the category of $\mathbf{F}_2$-vector spaces. We will refer to these objects simply as *complexes*. To each complex V we can associate cohomology groups

$$\mathrm{H}^n V = \ker(d_n)/\operatorname{Im}(d_{n-1}).$$

Remark 1. It is possible to take a more sophisticated point of view: we can identify cochain complexes V over the field $\mathbf{F}_2$ with *module spectra* over $\mathbf{F}_2$. The cohomology groups $\mathrm{H}^n(V)$ should then be viewed as the *homotopy groups* π_{-n} of the corresponding spectra.

Given a pair of $\mathbf{F}_2$-module spectra V and W, we can form their tensor product $V \otimes W$. This is given by the usual tensor product of complexes of vector spaces:

$$(V \otimes W)^n = \oplus_{n=n'+n''} V^{n'} \otimes W^{n''},$$

with the usual differential (note that, since we are working over the field $\mathbf{F}_2$, we do not even have to worry about signs). In particular, we can form the tensor powers

$$V^{\otimes n} = V \otimes V \otimes \ldots \otimes V$$

of a fixed $\mathbf{F}_2$-module spectrum. The tensor power $V^{\otimes n}$ inherits a natural action of the symmetric group Σ_n, by permuting the tensor factors.

One of the most important examples of an $\mathbf{F}_2$-module spectrum is the cochain complex

$$C^*(X; \mathbf{F}_2)$$

of a topological space X. The cohomology groups of this $\mathbf{F}_2$-module spectrum are simply the cohomology groups of X. The cohomology $\mathrm{H}^*(X; \mathbf{F}_2)$ has the structure of a graded commutative ring. The multiplication on $\mathrm{H}^*(X; \mathbf{F}_2)$ arises from a multiplication which exists on the cochain complex $C^*(X; \mathbf{F}_2)$. Namely, we can consider the composition

$$C^*(X; \mathbf{F}_2) \otimes C^*(X; \mathbf{F}_2) \rightarrow C^*(X \times X; \mathbf{F}_2) \rightarrow C^*(X; \mathbf{F}_2).$$

Here the first map is the classical Alexander-Whitney morphism, and the second is given by pullback along the diagonal inclusion $X \rightarrow X \times X$. The Alexander-Whitney map is *not* compatible with the action of the symmetric group Σ_2 on the two sides. Consequently, the resulting multiplication

$$m : C^*(X; \mathbf{F}_2) \otimes C^*(X; \mathbf{F}_2) \rightarrow C^*(X; \mathbf{F}_2)$$

is not commutative until passing to homotopy. The failure of m to be strictly commutative turns out to be a very interesting phenomenon, which is responsible for the existence of Steenrod operations.

In the above situation, the multiplication m is not commutative. However, it does induce a commutative multiplication after passing to cohomology. In fact, more is true: the map m satisfies a symmetry condition up to coherent homotopy. The following definitions allow us to make this idea precise:

Definition 2. Let V be an $\mathbf{F}_2$-module spectrum and $n \geq 0$ a nonnegative integer. The *nth extended power* of V is given by the homotopy coinvariants

$$V^{\otimes n}_{h\Sigma_n}.$$

This is a complex which we will denote by $D_n(V)$.

Remark 3. In concrete terms, $D_n(V)$ may be computed in the following way. Let M denote the vector space $\mathbf{F}_2$, with the trivial action of Σ_n. Choose a resolution

$$\ldots \to P^{-1} \to P^0 \to M$$

by free $\mathbf{F}_2[\Sigma_n]$-modules. We let $E\Sigma_n$ denote the complex $P^\bullet$. (We can think of $E\Sigma_n$ as a contractible complex with a free action of Σ_n.) The extended power $D_n(V)$ of a complex V can then be identified with the ordinary coinvariants

$$(V^{\otimes n} \otimes E\Sigma_n)_{\Sigma_n}.$$

Definition 4. Let V be a complex. A *symmetric multiplication* on V is a map

$$D_2(V) \to V.$$

Example 5. If X is any topological space, then the cochain complex $C^*(X; \mathbf{F}_2)$ can be endowed with a symmetric multiplication. If X is equipped with a base point $*$, then the reduced cochain complex $C^*(X, *; \mathbf{F}_2)$ also inherits a symmetric multiplication.

Example 6. Let X be an infinite loop space. Then the chain complex $C_*(X; \mathbf{F}_2)$ can be endowed with a symmetric multiplication.

Examples 5 and 6 are really special cases of the following:

Example 7. Let A be an E_∞-algebra over the field $\mathbf{F}_2$. Then A has an underlying $\mathbf{F}_2$-module spectrum, which is equipped with a symmetric multiplication.

Our goal in this lecture is to study the consequences of the existence of a symmetric multiplication on a complex V.

Notation 8. Let n be an integer. We let $\mathbf{F}_2[-n]$ denote the complex which consists of a 1-dimensional vector space in cohomological degree n, and zero elsewhere. Let e_n denote a generator for the $\mathbf{F}_2$-vector space $\mathrm{H}^n\, \mathbf{F}_2[-n]$, so we have isomorphisms

$$\mathrm{H}^k\, \mathbf{F}_2[-n] \simeq \begin{cases} \mathbf{F}_2 e_n & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

Our first goal is to describe the extended squares of complexes of the form $\mathbf{F}_2[-n]$. This is easy: we observe that $\mathbf{F}_2[-n]^{\otimes 2}$ is isomorphic to $\mathbf{F}_2[-2n]$, with the symmetric group Σ_2 acting trivially (since we are working in characteristic 2, there are no signs to worry about). Consequently, we can identify $D_2(\mathbf{F}_2[-n])$ with the tensor product

$$\mathbf{F}_2[-2n] \otimes (E\Sigma_2)_{\Sigma_2}.$$

The second tensor factor can be identified with the chain complex of the space $B\Sigma_2 \simeq \mathbf{R}P^\infty$. Consequently, we get canonical isomorphisms

$$\mathrm{H}^k(D_2(\mathbf{F}_2[-n]) \simeq \mathrm{H}_{2n-k}(B\Sigma_2; \mathbf{F}_2) e_{2n}.$$

We now recall the structure of the homology and cohomology of the space $B\Sigma_2 \simeq \mathbf{R}P^\infty$. There is a (unique) isomorphism

$$\mathrm{H}^*(\mathbf{R}P^\infty; \mathbf{F}_2) \simeq \mathbf{F}_2[t],$$

where the polynomial generator t lies in $\mathrm{H}^1(\mathbf{R}P^\infty; \mathbf{F}_2)$. We have a dual description of the homology $\mathrm{H}_*(\mathbf{R}P^\infty; \mathbf{F}_2)$: this is just a one-dimensional vector space in each degree m, with a unique generator which we will denote by x_m.

Definition 9. Let V be a complex, and let $v \in \mathrm{H}^n\, V$, so that v determines a homotopy class of maps

$$\eta : \mathbf{F}_2[-n] \to V.$$

For $i \leq n$, we let

$$\overline{\mathrm{Sq}}^i(v) \in \mathrm{H}^{n+i}\, D_2(V)$$

denote the image of

$$x_{n-i} \otimes e_{2n} \in \mathrm{H}_{n-i}(\mathbf{R}P^\infty; \mathbf{F}_2)e_{2n} \simeq \mathrm{H}^{n+i}\, D_2(\mathbf{F}_2[n])$$

under the induced map

$$D_2(\mathbf{F}_2[-n]) \stackrel{D_2(\eta)}{\to} D_2(V).$$

By convention, we will agree that $\overline{\mathrm{Sq}}^i(v) = 0$ for $i > n$.

If V is equipped with a symmetric multiplication $D_2(V) \to V$, we let $\mathrm{Sq}^i(v)$ denote the image of $\overline{\mathrm{Sq}}^i(v)$ under the induced map

$$\mathrm{H}^{n+i}\, D_2(V) \to \mathrm{H}^{n+i}\, V.$$

The operations $\mathrm{Sq}^i : \mathrm{H}^*\, V \to \mathrm{H}^{*+i}\, V$ are called the *Steenrod operations*, or *Steenrod squares*.

Example 10. Let V be an $\mathbf{F}_2$-module spectrum equipped with a symmetric multiplication, and let $v \in \mathrm{H}^n\, V$. Then $\mathrm{Sq}^n(v) \in \mathrm{H}^{2n}\, V$ is simply the image of $v \otimes v$ under the composite map

$$V \otimes V \to D_2(V) \to V.$$

In other words, Sq^n acts on $\mathrm{H}^n\, V$ by simply "squaring" the elements with respect to the multiplication on V. This is why the operations Sq^i are called "Steenrod squares".

Example 11. Let X be a topological space, and let $V = C^*(X; \mathbf{F}_2)$ be the cochain complex of X, equipped with its usual symmetric multiplication. Then Definition 9 yields operations

$$\mathrm{Sq}^i : \mathrm{H}^n(X; \mathbf{F}_2) \to \mathrm{H}^{n+i}(X; \mathbf{F}_2).$$

These are the usual Steenrod operations.

Remark 12. The operations $v \mapsto \overline{\mathrm{Sq}}^i\, v$ completely account for the cohomology groups of any extended square $D_2(V)$. More precisely, let us suppose that V is an $\mathbf{F}_2$-module spectrum, and that $\{v_i\}_{i \in I}$ is an ordered basis for $\pi_* V$, where $v_i \in \mathrm{H}^{n_i}\, V$. Then the collection

$$\{v_i v_j\}_{i<j} \cup \{\mathrm{Sq}^n\, v_i\}_{n \leq n_i}$$

is a basis for $\pi_* D_2(V)$. The proof of this is easy. Using the fact that D_2 commutes with filtered colimits, we can reduce to the case where only finitely many generators are involved. We then work by induction, using the formula

$$D_2(V \oplus W) \simeq (V \oplus W)^{\otimes 2}_{h\Sigma_2} \simeq V^{\otimes 2}_{h\Sigma_2} \oplus (V \otimes W) \oplus W^{\otimes 2}_{h\Sigma_2}$$

to reduce to the case of a single basis vector. The result is then obvious.

Proposition 13. *The Steenrod squares are* additive *operations. Let V be a complex, and let $v, v' \in \mathrm{H}^n V$. Then, for each integer k, we have*

$$\overline{\mathrm{Sq}}^k(v + v') = \overline{\mathrm{Sq}}^k(v) + \overline{\mathrm{Sq}}^k(v') \in \mathrm{H}^{n+k} D_2(V).$$

In particular, if V is equipped with a symmetric multiplication, we have

$$\mathrm{Sq}^k(v + v') = \mathrm{Sq}^k(v) + \mathrm{Sq}^k(v') \in \mathrm{H}^{n+k} V.$$

Proof. If $k > n$, then both sides are zero and there is nothing to prove. If $k = n$, then

$$\overline{\mathrm{Sq}}^k(v + v') = (v + v')^2 = \overline{\mathrm{Sq}}^k(v) + \overline{\mathrm{Sq}}^k(v') + (vv' + v'v).$$

Since the multiplication map

$$V \otimes V \to D_2(V)$$

is commutative on the level of homotopy, we have $vv' + v'v = 2vv' = 0$.

Now suppose that $k < n$. By functoriality, it will suffice to treat the universal case where $V \simeq \mathbf{F}[-n] \oplus \mathbf{F}[-n]$. Using Remark 12, we observe that the canonical map

$$\mathrm{H}^m D_2(V) \to \mathrm{H}^m D_2(\mathbf{F}_2[-n]) \times \mathrm{H}^m D_2(\mathbf{F}_2[-n])$$

is injective for $m < 2n$. We may therefore reduce to the case where either v or v' vanishes, in which case the result is obvious. □