LECTURE 11: HELP! WHITEHEAD THEOREM AND CELLULAR APPROXIMATION

1. HELP

The proof of the Whitehead theorem last time did not extend to infinite CW-complexes due to a technical difficulty with inverse limits. Rather that attempt to address this difficulty, we will instead use a more subtle homotopy extension property possessed by relative CW-complexes. This is the Homotopy Extension and Lifting Property (HELP):

Theorem 1.1 (HELP)**.** Suppose that (X, A) is a relative CW-complex of dimension $\leq n$ and that $f : Y \to Z$ is an n-equivalence (where n may be chosen to be ∞). Then for each diagram as below there exist compatible extensions and lifts

$$\begin{tikzcd}
& A\times\{1\} \arrow[r, "g"] \arrow[dl] \arrow[dd] & Y \arrow[dd, "f"] \\
X\times\{1\} \arrow[urr, dotted] \arrow[dd] & & \\
& A\times I\cup_{A\times\{0\}} X\times\{0\} \arrow[r] \arrow[dl] & Z \\
X\times I \arrow[urr, dotted] & &
\end{tikzcd}$$

In words: "for every map $g : A \to Y$ such that fg is homotopic to a map which extends to a map $\widetilde{g}$ over X, there is an extension $\widetilde{g}'$ of g over X, such that $f\widetilde{g}'$ is homotopic to $\widetilde{g}$, by a homotopy extending the original homotopy."

The proof of HELP is obtained by first considering the case $(X, A) = (D^n, S^{n-1})$ and then performing induction on the relative skeleta of (X, A).

Following May, the following Whitehead theorem may be deduced by clever application of HELP.

Theorem 1.2 (Whitehead theorem)**.** Suppose that Z is a CW-complex of dimension $< n \leq \infty$, and that $f : X \to Y$ is an n-equivalence. Then the induced map

$$[Z, X] \to [Z, Y]$$

is an isomorphism.

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2. Cellular Approximation

The Whitehead theorem implies that CW-complexes are good when dealing with weak equivalences. Not every space has the homotopy type of a CW-complex, however, every space is weakly equivalent to a CW-complex.

Theorem 2.1 (Cellular approximation)**.** Suppose that X is a space. Then there exists a CW-complex $\widetilde{X}$ together with a weak equivalence

$$\widetilde{X} \to X.$$

Proof. Assume X is path connected. Start by mapping in a wedge of spheres $\widetilde{X}^{[0]}$, one for every generator of every homotopy group of X. Then, inductively by degree k, attach reduced cylinders to $\widetilde{X}^{[k-1]}$ for every pair of elements of $\pi_k(\widetilde{X}^{[k-1]})$ which map to the same element in $\pi_k(X)$. Use the homotopies to produce a map $\widetilde{X}^{[k]} \to X$. which is a k-equivalence. $\square$

Remark 2.2. The complex $\widetilde{X}$ in the previous proof is technically not a CW complex, but rather a cell complex: the cell attachments are not ordered by dimension. This is OK: the attaching maps of k-cells factor up to homotopy through the $(k-1)$-skeleta.