LECTURE 4: SIMPLE COMPUTATIONS, THE ACTION OF THE FUNDAMENTAL GROUPOID

1. Simple computations

By using smooth or simplicial approximation, we can prove:

Proposition 1.1. For $k < n$, we have $\pi_k(S^n) = 0$.

Proposition 1.2. Suppose that $p_* : X \to Y$ is a covering space. Then the induced map
$$p_* : \pi_k(X) \to \pi_k(Y)$$
is a monomorphism for $k = 1$ and an isomorphism for $k > 1$.

Using the covering $\mathbb{R} \to S^1$, we have
$$\pi_k(S^1) = \begin{cases} \mathbb{Z}, & k = 1 \\ 0, & k \neq 1 \end{cases}$$
In fact, any space with a contractible universal cover has trivial higher homotopy groups. For instance, the torus T^2:
$$\pi_k(T^2) = \begin{cases} \mathbb{Z} \times \mathbb{Z}, & k = 1 \\ 0, & k \neq 1 \end{cases}$$
This result can be obtained in a different way using $T^2 = S^1 \times S^1$ together with the following lemma.

Lemma 1.3. There is an isomorphism $\pi_k(X \times Y) \cong \pi_k(X) \times \pi_k(Y)$.

In general we shall (eventually) see that $\pi_n(S^n) = \mathbb{Z}$. In general, the groups $\pi_k(S^n)$ for $k > n$ are a mess. They are known through a finite range, and are very difficult to compute.

Given a sequence of sets
$$T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} \cdots$$
we can form the *colimit* or *direct limit*:
$$\varinjlim T_i = \left(\coprod_i T_i \right) / (t \sim f_i(t)).$$

- If each of the f_i's are inclusions, then $\varinjlim T_i$ is the union.
- If the f_i's are continuous maps between topological spaces, then there is a natural quotient topology that can be placed on $\varinjlim T_i$.
- If each of the f_i's is a closed inclusion of compactly generated spaces, then $\varinjlim T_i$ is compactly generated, and has the topology of the union.
- If each of the f_i's is a homomorphism between groups, then $\varinjlim T_i$ is a group.

Date: 2/15/06.

Lemma 1.4. Let

$$X_0 \to X_1 \to X_2 \to \cdots$$

be a sequence of closed inclusions. Let K be a compact space. Then there is a bijection

$$\varinjlim \mathrm{Map}(K, X_i) \cong \mathrm{Map}(K, \varinjlim X_i).$$

Corollary 1.5. There is an isomorphism of groups

$$\varinjlim_i \pi_k(X_i) \cong \pi_k(\varinjlim_i X_i).$$

Let S^∞ be the colimit $\varinjlim_n S^n$. Proposition 1.1 and Corollary 1.5 combine to yield

$$\pi_k(S^\infty) = 0.$$

Since $S^\infty \to \mathbb{R}P^\infty$ is a two-fold cover, we have

$$\pi_k(\mathbb{R}P^\infty) = \begin{cases} \mathbb{Z}/2, & k = 1 \\ 0, & k \neq 1 \end{cases}$$

In general, if π is a group (abelian if $n > 1$), then if X is a space satisfying

$$\pi_k(X) = \begin{cases} \pi, & k = 1 \\ 0, & k \neq 1 \end{cases}$$

it is called an *Eilenberg-MacLane space* or a $K(\pi, n)$. Usually we assume also that X is a CW complex. We shall see that such spaces are unique up to homotopy.

2. The action of the fundamental groupoid

A *groupoid* is a category for which every morphism is an isomorphism. A group is a groupoid with one object: given a group G, we have a category with one object $*$, and morphisms $\mathrm{Map}(*, *) := G$. Group multiplication gives composition, and the existence of inverses implies that this category is in fact a groupoid.

Let X be an unpointed space. Define its fundamental groupoid $\pi_{oid}(X)$ to be the category whose objects are the points of X, and whose morphisms $x \to y$ are given by paths from x to y modulo homotopy relative to the endpoints. This is easily seen to be a groupoid

We showed that given a path $\gamma : x \to y$, we have an induced homomorphism

$$\gamma_* : \pi_k(X, x) \to \pi_k(X, y).$$

The homomorphism γ_* is easily seen to depend only on the class it represents in the fundamental groupoid.

Proposition 2.1. This action yields a functor

$$\begin{aligned} \pi_k(X, -) : \pi_{oid}(X) &\to \mathrm{Groups} \\ x &\mapsto \pi_k(X, x) \\ ([\gamma] : x \to y) &\mapsto (\gamma_* : \pi_k(X, x) \to \pi_k(X, y)). \end{aligned}$$

In particular, there is an action of $\pi_1(X, x)$ on $\pi_k(X, x)$.