LECTURE 1: CATEGORY THEORY

Category theory is very dry. It is nevertheless a useful language to speak of certain things.

1. Categories

Definition 1.1. A *category* consists of:

1. A collection of objects $Ob\mathcal{C}$.
2. For any pair of objects $x, y \in Ob\mathcal{C}$, a set of morphisms $\mathrm{Map}_{\mathcal{C}}(x, y)$.
3. For each x, an identity morphism $1_x \in \mathrm{Map}_{\mathcal{C}}(x, x)$.
4. For each x, y, z, a composition map

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) &\to \mathrm{Map}_{\mathcal{C}}(x, z) \\ (f, g) &\mapsto fg. \end{aligned}$$

These satisfy the following axioms:

- (Identity) for each $f \in \mathrm{Map}_{\mathcal{C}}(x, y)$, we have $1_y f = f 1_x = f$.
- (Associativity) for composible f, g, h, we have $(fg)h = f(gh)$.

Examples 1.2.

1. *Sets*: The category of sets, with morphisms all maps of sets.
2. *Gp*: The category of groups, with morphisms given by group homomorphisms.
3. *Ab*: The category of abelian groups, with morphisms given by group homomorphisms.
4. Mod_R: For R a ring, the category of R-modules, with morphisms the homomorphisms of R-modules.
5. *Top*: The category of topological spaces, with morphisms given by the continuous maps.
6. For G a group, we may associate to G a category $\underline{G}$. The category $\underline{G}$ has one object (call it $*$). The morphisms $\mathrm{Map}_{\underline{G}}(*, *)$ are given by the set G. The composition law is given by the group multiplication.

If $\mathcal{C}$ is a category, a morphism $f : x \to y$ is an *isomorphism* if there exists an inverse morphism $f^{-1} : y \to x$ such that $ff^{-1} = 1_y$ and $f^{-1}f = 1_x$.

2. Maps between categories: functors

Definition 2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *(covariant) functor* consists of a map

$$F : Ob\mathcal{C} \to Ob\mathcal{D}$$

together with maps

$$F : \mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(F(x), F(y))$$

for all objects x, y of $\mathcal{C}$, such that:

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(1) $F(1_x) = 1_{F(x)}$,
(2) $F(fg) = F(f)F(g)$.

The notion of a contravariant functor is identical, except the arrows are reversed.

Definition 2.2. A *(contravariant) functor* consists of a map

$$F : Ob\mathcal{C} \to Ob\mathcal{D}$$

together with maps

$$F : \mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(F(y), F(x))$$

for all objects x, y of $\mathcal{C}$, such that:

(1) $F(1_x) = 1_{F(x)}$,
(2) $F(fg) = F(g)F(f)$.

Given a category $\mathcal{C}$, we may form the opposite category $\mathcal{C}^{op}$ by keeping the objects the same, but reversing the direction of all of the morphisms:

$$\mathrm{Map}_{\mathcal{C}^{op}}(x, y) = \mathrm{Map}_{\mathcal{C}}(y, x).$$

Then there is a bijective correspondence:

$$\begin{gathered} \{\text{contravariant functors } \mathcal{C} \to \mathcal{D}\} \\ \updownarrow \\ \{\text{covariant functors } \mathcal{C}^{op} \to \mathcal{D}\} \end{gathered}$$

Examples 2.3.

(1) The forgetful functor

$$U : Gp \to Sets.$$

It associates to a group G the underlying set $U(G)$ (forget the group multiplication).

(2) The free group functor

$$F : Sets \to Gp.$$

It associates to a set S the free group $F(S)$ generated by S.

(3) Let A be an object of Ab, the category of abelian groups. Then

$$\begin{aligned} \mathrm{Hom}(A, -) : Ab &\to Ab \\ B &\mapsto \mathrm{Hom}(A, B) \end{aligned}$$

is a covariant functor (why?). Dually,

$$\begin{aligned} \mathrm{Hom}(-, A) : Ab &\to Ab \\ B &\mapsto \mathrm{Hom}(B, A) \end{aligned}$$

is a contravariant functor (again, why?). In fact, we may regard $\mathrm{Hom}(-, -)$ as giving a covariant functor

$$Ab^{op} \times Ab \to Ab$$

where the category $Ab^{op} \times Ab$ is the product category. It has objects given as pairs of objects, and morphisms given by pairs of morphisms.

(4) The association $X \mapsto \pi_1(X)$ gives a functor
$$\pi_1 : Top_* \to Gp$$
(here Top_* is the category of based spaces).

(5) The association $X \mapsto H^k(X; R)$ gives a contravariant functor
$$H^k(-; R) : Top \to Mod_R$$
(here R is a ring).

If you want to think about functors as maps between categories, then you should think about natural transformations as maps between functors, that is, maps between maps between categories!

Definition 2.4. Let
$$F, G : \mathcal{C} \to \mathcal{D}$$
be a pair of (contravariant) functors. A *natural transformation*
$$\eta : F \to G$$
consists of a collection of morphisms in $\mathcal{D}$
$$\eta_x : F(x) \to G(x)$$
for each $x \in \mathcal{C}$. These morphisms must satisfy the following *naturality condition*: for each morphism $f : x \to y$ in $\mathcal{C}$, we require that the following diagram commute:
$$\begin{tikzcd} F(x) \arrow[r, "\eta_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "\eta_y"'] & G(y) \end{tikzcd}$$

Example 2.5. Let A, B be an objects of Ab, and consider the contravariant functors
$$\mathrm{Hom}(-, A), \mathrm{Hom}(-, B) : Ab \to Ab.$$
Then a homomorphism $f : A \to B$ gives rise to a natural transformation
$$f_* : \mathrm{Hom}(-, A) \to \mathrm{Hom}(-, B).$$
It is given by the maps
$$f_* : \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$$
$$g \mapsto fg.$$