19 The Kronecker-Weber theorem

As you proved in Problem Set 4, for each integer $m > 1$ the cyclotomic extension $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ is an abelian extension with Galois group $G := \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$. If K is a subfield of $\mathbb{Q}(\zeta_m)$, then the subgroup H of G fixing K is necessarily normal (since G is abelian), thus $K/\mathbb{Q}$ is Galois, with $\mathrm{Gal}(K/\mathbb{Q}) \simeq G/H$, which we note is also abelian. We thus have a simple recipe for constructing finite abelian extensions of $\mathbb{Q}$: pick $m \geq 1$ and take any subfield of $\mathbb{Q}(\zeta_m)$.

Remarkably, every finite abelian extension of $\mathbb{Q}$ can be constructed in this way. This is the *Kronecker-Weber Theorem*, which was first stated by Kronecker [2] in 1853. Kronecker proved it for extensions of odd degree and Weber published a proof 1886 [5] that was believed to address the remaining cases; in fact Weber's proof contains some gaps (as noted in [3]), but in any case an alternative proof was given a few years later by Hilbert [1].

The proof of the Kronecker-Weber theorem we present here is adapted from [4, Ch. 14]

19.1 Local and global Kronecker-Weber theorems

We now state the (global) Kronecker-Weber theorem.

Theorem 19.1. *Every finite abelian extension of $\mathbb{Q}$ lies in a cyclotomic field $\mathbb{Q}(\zeta_m)$.*

There is also a local version.

Theorem 19.2. *Every finite abelian extension of $\mathbb{Q}_p$ lies in a cyclotomic field $\mathbb{Q}_p(\zeta_m)$.*

In fact, the local and global versions are equivalent.

Proposition 19.3. *The global Kronecker-Weber theorem holds if and only if the local Kronecker-Weber theorem holds.*

Proof. If $\hat{K}/\mathbb{Q}_p$ is a finite abelian extension of local fields, then, by Corollary 11.3, there is a corresponding Galois extension $K/\mathbb{Q}$ of global fields such that $\hat{K}$ is the completion of K with respect to a $\mathfrak{p}$-adic absolute value extending the p-adic absolute value on $\mathbb{Q}$. The Galois group $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathrm{Gal}(\hat{K}/\mathbb{Q}_p)$ is abelian, so the global Kronecker-Weber theorem implies that $K \subseteq \mathbb{Q}(\zeta_m)$ for some integer $m > 1$. Let $\hat{L}$ be the completion of $\mathbb{Q}(\zeta_m)$ at prime $\mathfrak{q}|\mathfrak{p}$. Then $\hat{L}$ contains $\mathbb{Q}_p(\zeta_m)$, and since $\mathbb{Q}_p(\zeta_m)$ is a complete field containing $\mathbb{Q}(\zeta_m)$ the two fields must be equal. Thus $\hat{K} \subseteq \hat{L} \subseteq \mathbb{Q}_p(\zeta_m)$, so the local Kronecker-Weber theorem holds.

Now let $K/\mathbb{Q}$ be a finite abelian extension of global fields. For each ramified prime p of $\mathbb{Q}$, pick a prime $\mathfrak{p}|p$ and let $K_\mathfrak{p}$ be the completion of K at $\mathfrak{p}$. The extension $K_\mathfrak{p}/\mathbb{Q}_p$ is finite abelian (its Galois group is isomorphic to a subgroup of $\mathrm{Gal}(K/\mathbb{Q})$, by part (6) of Theorem 11.4), and the local Kronecker-Weber theorem implies $K_\mathfrak{p} \subseteq \mathbb{Q}_p(\zeta_{m_p})$ for some integer $m_p \geq 1$. Now let $e_p = v_p(m_p)$ and define $m := \prod_p p^{e_p}$ (this is a finite product, since it ranges over ramified primes).

Claim: $K(\zeta_m) = \mathbb{Q}(\zeta_m)$ (and in particular, $K \subseteq \mathbb{Q}(\zeta_m)$).

Proof of claim: Let $L = K(\zeta_m)$. Then L is Galois (it is the splitting field over K of the cyclotomic polynomial $\Phi_m(x)$), and it is abelian since its Galois group is isomorphic to a subgroup of $\mathrm{Gal}(K/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ (because $L = K \cdot \mathbb{Q}(\zeta_m)$). Let $\mathfrak{q}$ be a prime of L lying above one of our chosen $\mathfrak{p}|p$; then $\mathfrak{q}|p$ and the completion $L_\mathfrak{q}$ of L at $\mathfrak{q}$ is a finite abelian extension of $\mathbb{Q}_p$. Let F be the maximal unramified extension of $\mathbb{Q}_p$ in $L_\mathfrak{q}$. Then $L_\mathfrak{q}/F$ is totally ramified, so its Galois group is isomorphic to the inertia group $I_p := I_\mathfrak{q}$. The field F

contains roots of unity ζ_n for all $n|m$ not divisible by p (because the extensions $\mathbb{Q}_p(\zeta_n)$ are all unramified and F is maximal), so $L_{\mathfrak{q}} = F(\zeta_m) = F(\zeta_{p^{e_p}})$. Note that $F \cap \mathbb{Q}(\zeta_{p^{e_p}}) = \mathbb{Q}_p$, since the extension $\mathbb{Q}_p(\zeta_{p^{e_p}})/\mathbb{Q}_p$ must be ramified if its nontrivial, and therefore

$$I_p \simeq \mathrm{Gal}(L/F) \simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^{e_p}})) \simeq (\mathbb{Z}/p^{e_p}\mathbb{Z})^\times.$$

Now let I be the subgroup of $\mathrm{Gal}(L/\mathbb{Q})$ generated by the inertia groups I_p for $p|m$. Then

$$\#I \leq \prod_p \#I_p = \prod_p \phi(p^{e_p}) = \phi(m) = [\mathbb{Q}(\zeta_m) : \mathbb{Q}].$$

The fixed field of I is an unramified extension of $\mathbb{Q}$, hence trivial (by Corollary 13.23). Therefore $I = \mathrm{Gal}(L/\mathbb{Q})$ and

$$[L : \mathbb{Q}] = \#I \leq [\mathbb{Q}(\zeta_m) : \mathbb{Q}],$$

so $K(\zeta_m) = L = \mathbb{Q}(\zeta_m)$ and the global Kronecker-Weber theorem holds for $K \subseteq \mathbb{Q}(\zeta_m)$. $\square$

To prove the local Kronecker-Weber theorem we first reduce to the case of cyclic extensions of prime-power degree. Recall that if L_1 and L_2 are two Galois extensions of a field K then compositum $L = L_1L_2$ is Galois over K and

$$\mathrm{Gal}(L/K) \simeq \{(\sigma_1, \sigma_2) : \sigma_1|_{L_1\cap L_2} = \sigma_2|_{L_1\cap L_2}\} \subseteq \mathrm{Gal}(L_1/K) \times \mathrm{Gal}(L_2/K).$$

Note that the inclusion on the RHS is an equality if and only if $L_1 \cap L_2 = K$. If L/K is an abelian extension with $\mathrm{Gal}(L/K) \simeq H_1 \times H_2$ then by defining $L_2 := L^{H_1}$ and $L_1 := L^{H_2}$ we may write $L = L_1L_2$ with $L_1 \cap L_2 = K$, and we then have $\mathrm{Gal}(L_1/K) \simeq H_1$ and $\mathrm{Gal}(L_2/K) \simeq H_2$. It then follows from the structure theorem for finite abelian groups that we may decompose any finite abelian extension L/K into a compositum $L = L_1 \cdots L_n$ of (linearly disjoint) cyclic extensions L_i/K of prime-power degree. If each L_i lies in $K(\zeta_{m_i})$ for some integer $m_i \geq 1$, then if we put $m := m_1 \cdots m_n$ we have $L \subseteq \mathbb{Q}(\zeta_m)$.

To prove the local Kronecker-Weber theorem it suffices to consider cyclic ℓ-extensions $K/\mathbb{Q}_p$ (cyclic extensions whose degree is a power of a prime ℓ). There two distinct cases: $\ell = p$ and $\ell \neq p$. We consider the easier case $\ell \neq p$ first.

19.2 The Kronecker-Weber theorem for cyclic ℓ-extensions of $\mathbb{Q}_p$with $\ell \neq p$

Proposition 19.4. *Let $K/\mathbb{Q}_p$ be a cyclic extension of degree ℓ^r for some prime $\ell \neq p$. Then $K \subseteq \mathbb{Q}_p(\zeta_m)$ for some $m \in \mathbb{Z}_{\geq 1}$.*

Proof. Let F be the maximal unramified extension of $\mathbb{Q}_p$ in K; then F is cyclotomic, by Corollary 10.5, so let $F = \mathbb{Q}_p(\zeta_n)$. The extension K/F is totally ramified, and it must be tamely ramified, since the ramification index is necessarily a power of ℓ and therefore not divisible by p. By Theorem 10.23, we have $K = F(\pi^{1/e})$ for some uniformizer π of the discrete valuation ring $\mathcal{O}_F$, with $e = [K : F]$. We may assume that $\pi = -pu$ for some $u \in \mathcal{O}_F^\times$, since $F/\mathbb{Q}_p$ is unramified: if $\mathfrak{q}|p$ is the maximal ideal of $\mathcal{O}_F$ then the valuation $v_{\mathfrak{q}}$ extends v_p with index $e_{\mathfrak{q}} = 1$ (by Theorem 5.11), so $v_{\mathfrak{q}}(-pu) = v_p(-pu) = 1$. The field $K = F(\pi^{1/e})$ then lies in the compositum of $F((-p)^{1/e})$ and $F(u^{1/e})$, and we will show that both of these fields lie in a cyclotomic extension of $\mathbb{Q}_p$.

The extension $F(u^{1/e})/F$ is unramified, since $p \nmid e$ and u is a unit (the discriminant of $x^e - u$ is not divisible by p), thus $F(u^{1/e})/\mathbb{Q}_p$ is unramified and therefore cyclotomic, by Corollary 10.5, so let $F(u^{1/e}) = \mathbb{Q}_p(\zeta_k)$ for some integer $k \geq 1$. The field $K(u^{1/e}) = K \cdot \mathbb{Q}_p(\zeta_k)$ is a compositum of abelian extensions, so $K(u^{1/e})/\mathbb{Q}_p$ is abelian, and it contains the subextension $\mathbb{Q}_p((-p)^{1/e})/\mathbb{Q}_p$, which must be Galois (since it lies in an abelian extension) and totally ramified (by Theorem 10.18, since it is an Eisenstein extension). The field $\mathbb{Q}_p((-p)^{1/e})$ contains ζ_e (take ratios of roots of $x^e + p$) and is totally ramified (since it is Eisenstein), but $\mathbb{Q}_p(\zeta_e)/\mathbb{Q}_p$ is unramified (since $p \nmid e$), so we must have $\mathbb{Q}_p(\zeta_e) = \mathbb{Q}_p$. Therefore $e|(p-1)$, and by Lemma 19.5 below we have

$$\mathbb{Q}_p((-p)^{1/e}) \subseteq \mathbb{Q}_p((-p)^{1/(p-1)}) = \mathbb{Q}_p(\zeta_p),$$

It follows that $F((-p)^{1/e}) = F \cdot \mathbb{Q}_p((-p)^{1/e}) \subseteq \mathbb{Q}_p(\zeta_n) \cdot \mathbb{Q}_p(\zeta_p)$. If we now put $m = npk$, the cyclotomic field $\mathbb{Q}_p(\zeta_m)$ contains both $F(u^{1/e})$ and $F((-p)^{1/e})$, and therefore K. $\square$

Lemma 19.5. *For any prime p we have $\mathbb{Q}_p\big((-p)^{1/(p-1)}\big) = \mathbb{Q}_p(\zeta_p)$.*

Proof. Let $\alpha = (-p)^{1/(p-1)}$. Then α is a root of the Eisenstein polynomial $x^{p-1} + p$, so the extension $\mathbb{Q}_p((-p)^{1/(p-1)}) = \mathbb{Q}_p(\alpha)$ is totally ramified of degree $p-1$, and α is a uniformizer (by Proposition 10.17 and Theorem 10.18). Let $\pi = \zeta_p - 1$. The minimal polynomial of π is

$$f(x) := \frac{(x+1)^p - 1}{x} = x^{p-1} + px^{p-2} + \cdots + p,$$

which is Eisenstein, so $\mathbb{Q}_p(\pi) = \mathbb{Q}_p(\zeta_p)$ is also totally ramified of degree $p-1$, and π is a uniformizer. We have $u := -\pi^{p-1}/p \equiv 1 \bmod \pi$, so u is a unit in the ring of integers of $\mathbb{Q}_p(\zeta_p)$. If we now put $g(x) = x^{p-1} - u$ then $g(1) \equiv 0 \bmod \pi$ and $g'(1) = p - 1 \not\equiv 0 \bmod \pi$, so by Hensel's Lemma 9.13 we can lift 1 to a root β of $g(x)$ in $\mathbb{Q}_p(\zeta_p)$.

We then have $p\beta^{p-1} = pu = -\pi^{p-1}$, so $(\pi/\beta)^{p-1} + p = 0$, and therefore $\pi/\beta \in \mathbb{Q}_p(\zeta_p)$ is a root of the minimal polynomial of α. Since $\mathbb{Q}_p(\zeta_p)$ is Galois, this implies that $\alpha \in \mathbb{Q}_p(\zeta_p)$, and since $\mathbb{Q}_p(\alpha)$ and $\mathbb{Q}_p(\zeta_p)$ both have degree $p-1$, the two fields must be equal. $\square$

To complete the proof of the local Kronecker-Weber theorem, we need to address the case $\ell = p$, that is, we need to show that every cyclic p-extension of $\mathbb{Q}_p$ lies in a cyclotomic field. Here we need to deal with wild ramification, which complicates matters. We first recall a bit of the theory of Kummer extensions.

19.3 A little Kummer theory

Let K be a field, let $n \geq 1$ be prime to the characteristic of K, and assume K contains a primitive nth root of unity ζ_n. If L/K is an extension of the form $L = K(\sqrt[n]{a})$, then L is the splitting field of $f(x) = x^n - a$ over K (the roots $\zeta_n^i \alpha$ of $f(x)$ all lie in L), hence Galois; here $\sqrt[n]{a}$ denotes a root of $x^n - a$, but since L contains all of them, it makes no difference which one we pick. The extension L/K is cyclic, since we have an injective homomorphism

$$\begin{aligned} \mathrm{Gal}(L/K) &\hookrightarrow \langle \zeta_n \rangle \simeq \mathbb{Z}/n\mathbb{Z} \\ \sigma &\mapsto \frac{\sigma(\sqrt[n]{a})}{\sqrt[n]{a}}, \end{aligned}$$

which is an isomorphism whenever $x^n - a$ is irreducible.

Kummer's key observation is that the converse holds.

Lemma 19.6. *Let K be a field, let $n \geq 1$ be prime to the characteristic of K, and assume $\zeta_n \in K$. If L/K is a cyclic extension of degree n then $L = K(\sqrt[n]{a})$ for some $a \in K$.*

Proof. Let L/K be a cyclic extension of degree n with $\mathrm{Gal}(L/K) = \langle \sigma \rangle$. Applying Hilbert's Theorem 90 (Lemma 19.7 below) to ζ_n with $\mathrm{N}_{L/K}(\zeta_n) = \zeta_n^n = 1$, we obtain an element $\alpha \in L$ for which $\sigma(\alpha) = \zeta_n \alpha$. We have

$$\sigma(\alpha^n) = \sigma(\alpha)^n = (\zeta_n \alpha)^n = \alpha^n,$$

thus $a = \alpha^n$ is invariant under the action of $\langle \sigma \rangle = \mathrm{Gal}(L/K)$ and therefore lies in K. Moreover, the orbit $\{\alpha, \zeta_n \alpha, \ldots, \zeta_n^{n-1}\alpha\}$ of α under the action of $\mathrm{Gal}(L/K)$ has order n, so $L = K(\alpha) = K(\sqrt[n]{a})$ as desired. $\square$

Lemma 19.7 (Hilbert Theorem 90)**.** *Let L/K be a cyclic extension with Galois group $\langle \sigma \rangle$. For every $u \in L$ of norm $\mathrm{N}_{L/K}(u) = 1$ there exists $z \in L^\times$ for which $\sigma(z) = uz$.*

Proof. By the normal basis theorem, we can pick $b \in L$ so that $\{\sigma^i(b)\}$ is a basis for $L \simeq K^n$ as a K-vector space. If we represent elements of L in this basis, σ acts as a cyclic permutation $(x_1, \ldots, x_n) \mapsto (x_n, x_1, \ldots, x_{n-1})$. The map $f(x) = \sigma(ux)$ is a K-linear transformation of L, and we claim that 1 is an eigenvalue of f, a property that is invariant under base change. If we base-change to L, our n-dimensional K-vector space $L \simeq K^n$ becomes an n-dimensional L-vector space $L \otimes_K L \simeq L^n$, and the nonzero vector

$$(1,\, \sigma(u),\, \sigma(u)\sigma^2(u),\, \ldots,\, \sigma(u)\sigma^2(u)\sigma^3(u)\cdots\sigma^{n-1}(u)) \in L^n$$

is fixed by f (because $\sigma(u)\sigma^2(u)\cdots\sigma^{n-1}(u) = \mathrm{N}_{L/K}(u)u^{-1} = u^{-1}$). Thus 1 is an eigenvalue of f, so there is a nonzero $z \in L \simeq K^n$ that is fixed by f. $\square$

Definition 19.8. Let K be a field with algebraic closure $\overline{K}$, let $n \geq 1$ be prime to the characteristic of K, and assume $\zeta_n \in K$. The *Kummer pairing* is the map

$$\begin{aligned} \langle \cdot, \cdot \rangle \colon \mathrm{Gal}(\overline{K}/K) \times K^\times &\to \langle \zeta_n \rangle \\ \langle \sigma, a \rangle &\mapsto \sigma(\alpha)/\alpha \end{aligned}$$

where α is any nth root of a in $\in \overline{K}^\times$; if β is another nth root of a, then $\alpha/\beta \in K$ is fixed by σ (since K contains all nth roots of 1) and $\sigma(\beta)/\beta = \sigma(\beta)/\beta \cdot \sigma(\alpha/\beta)/(\alpha/\beta) = \sigma(\alpha)/\alpha$, so the value of $\langle \sigma, a \rangle$ does not depend on the choice of α. Note that if $a \in K^{\times n}$ then $\langle \sigma, a \rangle = 1$ for all $\sigma \in \mathrm{Gal}(\overline{K}, K)$, so the Kummer pairing depends only on the image of a in $K^\times/K^{\times n}$; thus we may also view it as a pairing on $\mathrm{Gal}(\overline{K}, K) \times K^\times/K^{\times n}$.

Theorem 19.9. *Let K be a field, let $n \geq 1$ be prime to the characteristic of K with $\zeta_n \in K$. The Kummer pairing induces an isomorphism*

$$\begin{aligned} \Phi \colon K^\times/K^{\times n} &\to \mathrm{Hom}\big(\mathrm{Gal}(\overline{K}/K), \langle \zeta_n \rangle\big) \\ a &\mapsto \big(\sigma \mapsto \langle \sigma, a \rangle\big). \end{aligned}$$

Proof. For each $a \in K^\times - K^{\times n}$, if we pick an nth root $\alpha \in \overline{K}$ of a then the extension $K(\alpha)/K$ will be non-trivial and some $\sigma \in \mathrm{Gal}(\overline{K}/K)$ must act nontrivially on α. For this σ we have $\langle \sigma, a \rangle \neq 1$, so the homomorphism $\Phi(a)$ is nontrivial and $a \notin \ker \Phi$. This shows that Φ is injective.

To show surjectivity, let $f\colon \mathrm{Gal}(\overline{K}/K) \to \langle\zeta_n\rangle$ be a homomorphism, let $d = \#\,\mathrm{im}\, f$, let $H = \ker f$, and let $L = \overline{K}^H$. Then $\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(\overline{K}/K)/H \simeq \mathbb{Z}/d\mathbb{Z}$, so L/K is a cyclic extension of degree d, and Lemma 19.6 implies that $L = K(\sqrt[d]{a})$ for some $a \in K$. If we put $e = n/d$ and consider the homomorphisms $\Phi(a^{me})$ for $m \in (\mathbb{Z}/d\mathbb{Z})^\times$, these homomorphisms are all distinct (because the a^{me} are distinct modulo $K^{\times n}$ and Φ is injective) and they all have the same kernel and image as f (their kernels have the same fixed field L because L contains all the dth roots of a). There are $\#(\mathbb{Z}/d\mathbb{Z})^\times = \#\mathrm{Aut}(\mathbb{Z}/d\mathbb{Z})$ distinct isomorphisms $\mathrm{Gal}(\overline{K}/K)/H \simeq \mathbb{Z}/d\mathbb{Z}$, one of which corresponds to f, and each corresponds to one of the $\Phi(a^{me})$. It follows that $f = \Phi(a^{me})$ for some $m \in (\mathbb{Z}/d\mathbb{Z})^\times$, so Φ is surjective. □

If we now consider any finite subgroup A of $K^\times/K^{\times n}$, we can choose $a_1, \ldots, a_r \in K^\times$ so that the images $\bar{a}_i$ of the a_i in $K^\times/K^{\times n}$ form a basis for the abelian group A; this means

$$A = \langle\bar{a}_1\rangle \times \cdots \times \langle\bar{a}_r\rangle \simeq \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_r\mathbb{Z},$$

where $n_i|n$ is the order of a_i in A. For each a_i, the fixed field of the kernel of $\Phi(a_i)$ is a cyclic extension of K isomorphic to $L_i := K(\sqrt[n_i]{a_i})$, as in the proof of Theorem 19.9. The fields L_i are linearly disjoint over K (because the a_i correspond to independent generators of A), and their compositum $L = K(\sqrt[n_1]{a_1}, \ldots \sqrt[n_r]{a_r})$ has Galois group $\mathrm{Gal}(L/K) \simeq A$, an abelian group whose exponent divides n; such fields L are called *n-Kummer extensions* of K (assuming $\zeta_n \in K$).

Conversely, given an n-Kummer extension L/K, we can iteratively apply Lemma 19.6 to put L in the form $L = K(\sqrt[n_1]{a_1}, \ldots, \sqrt[n_r]{a_r})$ with each $a_i \in K^\times$ and $n_i|n$, and the images of the a_i in $K^\times/K^{\times n}$ generate a subgroup A corresponding to L. We thus have a 1-to-1 correspondence between finite subgroups of $K^\times/K^{\times n}$ and (finite) n-Kummer extensions of K (this correspondence also extends to infinite subgroups provided we put a suitable topology on the groups).

So far we have been assuming that K contains all the nth roots of unity. To help handle situations where this is not necessarily the case, we rely on the following lemma, in which we restrict to the case that n is a prime (or an odd prime power) so that $(\mathbb{Z}/n\mathbb{Z})^\times$ is cyclic (the definition of ω in the statement of the lemma does not make sense otherwise).

Lemma 19.10. *Let n be a prime (or an odd prime power), let F be a field of characteristic prime to n, let $K = F(\zeta_n)$, and let $L = K(\sqrt[n]{a})$ for some $a \in K^\times$. Define the homomorphism $\omega\colon \mathrm{Gal}(K/F) \to (\mathbb{Z}/n\mathbb{Z})^\times$ by $\zeta_n^{\omega(\sigma)} = \sigma(\zeta_n)$. If L/F is abelian then $\sigma(a)/a^{\omega(\sigma)} \in K^{\times n}$ for all $\sigma \in \mathrm{Gal}(K/F)$.*

Proof. Let $G = \mathrm{Gal}(L/F)$, let $H = \mathrm{Gal}(L/K) \subseteq G$, and let A be the subgroup of $K^\times/K^{\times n}$ generated by a. The Kummer pairing induces a bilinear pairing $H \times A \to \langle\zeta_n\rangle$ that is compatible with the action of $\mathrm{Gal}(K/F) \simeq G/H$. In particular, we have

$$\langle h, a^{\omega(\sigma)}\rangle = \langle h, a\rangle^{\omega(\sigma)} = \sigma(\langle h, a\rangle) = \langle\sigma(h), \sigma(a)\rangle = \langle h, \sigma(a)\rangle$$

for all $\sigma \in \mathrm{Gal}(K/F)$ and $h \in H$; the Galois action on H is by conjugation (lift σ to G and conjugate there), but it is trivial because G is abelian. The pairing is nondegenerate (because Φ is injective), so we must have $a^{\omega(\sigma)} \equiv \sigma(a) \bmod K^{\times n}$; the lemma follows. □

19.4 The Kronecker-Weber theorem for cyclic p-extensions of $\mathbb{Q}_p$, for $p > 2$

We are now ready to prove the local Kronecker-Weber theorem in the case $\ell = p$. We first consider the case $p \neq 2$.

Theorem 19.11. *Let $p \neq 2$ be prime and let $K/\mathbb{Q}_p$ be a cyclic extension of degree p^r. Then $K \subseteq \mathbb{Q}_p(\zeta_m)$ for some $m \geq 1$.*

Proof. There are two obvious candidates for K, namely, the cyclotomic field $\mathbb{Q}_p(\zeta_{p^{p^r}-1})$, which by Corollary 10.5 is an unramified extension of degree p^r, and the index $p-1$ subfield of the cyclotomic field $\mathbb{Q}_p(\zeta_{p^{r+1}})$, which is a totally ramified extension of degree p^r (the p^{r+1}-cyclotomic polynomial has degree $p^r(p-1)$ and is irreducible over $\mathbb{Q}_p$). If K is contained in the compositum of these two fields then $K \subseteq \mathbb{Q}_p(\zeta_m)$, where $m := (p^{p^r}-1)(p^{r+1})$ and the theorem holds. Otherwise, the field $K(\zeta_m)$ is a Galois extension of $\mathbb{Q}_p$ with

$$\operatorname{Gal}(K(\zeta_m)/\mathbb{Q}_p) \simeq \mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/p^s\mathbb{Z},$$

for some $s > 0$; the first factor comes from the Galois group of $\mathbb{Q}_p(\zeta_{p^{p^r}-1})$, the second two factors come from the Galois group of $\mathbb{Q}_p(\zeta_{p^{r+1}})$ (note that $\mathbb{Q}_p(\zeta_{p^{r+1}}) \cap \mathbb{Q}_p(\zeta_{p^{p^r}-1}) = \mathbb{Q}_p$), and the last factor comes from the fact that we are assuming $K \not\subseteq \mathbb{Q}_p(\zeta_m)$, so $\operatorname{Gal}(K(\zeta_m)/\mathbb{Q}_p(\zeta_m))$ is nontrivial and must have order p^s for some $0 < s \leq r$.

It follows that the abelian group $\operatorname{Gal}(K(\zeta_m)/\mathbb{Q}_p)$ has a quotient isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$, and the subfield of $K(\zeta_m)$ corresponding to this quotient is an abelian extension of $\mathbb{Q}_p$ with Galois group isomorphic $(\mathbb{Z}/p\mathbb{Z})^3$. But by Lemma 19.12 below, no such field exists. $\square$

Lemma 19.12. *For $p > 2$ no extension of $\mathbb{Q}_p$ has Galois group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$.*

Proof. Suppose for the sake of contradiction that K is an extension of $\mathbb{Q}_p$ with Galois group $\operatorname{Gal}(K/\mathbb{Q}_p) \simeq (\mathbb{Z}/p\mathbb{Z})^3$. Then $K/\mathbb{Q}_p$ is linearly disjoint from $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$, since the order of $G := \operatorname{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p) \simeq (\mathbb{Z}/p\mathbb{Z})^\times$ is not divisible by p, and $\operatorname{Gal}(K(\zeta_p)/\mathbb{Q}_p(\zeta_p)) \simeq (\mathbb{Z}/p\mathbb{Z})^3$ is a p-Kummer extension. There is thus a subgroup $A \subseteq \mathbb{Q}_p(\zeta_p)^\times/\mathbb{Q}_p(\zeta_p)^{\times p}$ isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$, for which $K(\zeta_p) = \mathbb{Q}_p(\zeta_p, A^{1/p})$, where $A^{1/p} := \{a^{1/p} : a \in A\}$ (here we identify elements of A by representatives in $\mathbb{Q}_p(\zeta_p)^\times$ that are determined only up to pth powers).

For any $a \in A$, the extension $\mathbb{Q}_p(\zeta_p, \sqrt[p]{a})/\mathbb{Q}_p$ is abelian, so by Lemma 19.10, we have

$$\sigma(a)/a^{\omega(\sigma)} \in \mathbb{Q}_p(\zeta_p)^{\times p} \tag{1}$$

for all $\sigma \in G$, where $\omega\colon G \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^\times$ is the isomorphism defined by $\sigma(\zeta_p) = \zeta_p^{\omega(\sigma)}$.

We may take $\pi = \zeta_p - 1$ as a uniformizer for $\mathbb{Q}_p(\zeta_p)$, which we note is a totally ramified extension of $\mathbb{Q}_p$ of degree $p-1$ with residue field $\mathbb{Z}/p\mathbb{Z}$ (see the proof of Lemma 19.5; note that a totally ramified extension must have residue field degree 1). For each $a \in A$ we have

$$v_\pi(a) = v_\pi(\sigma(a)) \equiv \omega(\sigma)v_\pi(a) \bmod p,$$

thus $(1-\omega(\sigma))v_\pi(a) \equiv 0 \bmod p$, for all $\sigma \in G$, hence for all $\omega(\sigma) \in \omega(G) = (\mathbb{Z}/p\mathbb{Z})^\times$; since $p > 2$, this implies $v_\pi(a) \equiv 0 \bmod p$. Now a is determined only up to pth-powers, so after multiplying by $\pi^{-v_\pi(a)}$ we may assume $v_\pi(a) = 0$, and after multiplying by a suitable power of $\zeta_{p-1}^p = \zeta_{p-1}$, we may assume $a \equiv 1 \bmod \pi$, since the image of ζ_{p-1} generates the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ of the residue field.

We may thus assume that $A \subseteq U_1/U_1^p$, where $U_1 := \{u \equiv 1 \bmod \pi\}$. Each $u \in U_1$ can be written as a power series in π with integer coefficients in $[0, p-1]$ and constant coefficient 1.

We have $\zeta_p \in U_1$, since $\zeta_p = 1 + \pi$, and $\zeta_p^b = 1 + b\pi + O(\pi^2)$ for $b \in [0, p-1]$.[1] Thus for any $a \in A \subseteq U_1$, we can choose b so that for some $c \in \mathbb{Z}$ and $e \in \mathbb{Z}_{\geq 2}$ we have

$$a = \zeta_p^b(1 + c\pi^e + O(\pi^{e+1})).$$

[1] The expression $O(\pi^n)$ denotes a power series in π that is divisible by π^n.

For $\sigma \in G$ we have

$$\frac{\sigma(\pi)}{\pi} = \frac{\sigma(\zeta_p - 1)}{\zeta_p - 1} = \frac{\zeta_p^{\omega(\sigma)} - 1}{\zeta_p - 1} = \zeta_p^{\omega(\sigma)-1} + \cdots + \zeta_p + 1 \equiv \omega(\sigma) \bmod \pi,$$

since each term in the sum is congruent to 1 modulo $\pi = (\zeta_p - 1)$; here we are representing $\omega(\sigma) \in (\mathbb{Z}/p\mathbb{Z})^\times$ as an integer in $[1, p-1]$. Thus $\sigma(\pi) \equiv \omega(\sigma)\pi \bmod \pi$ and

$$\sigma(a) = \zeta_p^{b\omega(\sigma)}(1 + c\omega(\sigma)^e\pi^e + O(\pi^{e+1})).$$

We also have

$$a^{\omega(\sigma)} = \zeta_p^{b\omega(\sigma)}(1 + c\omega(\sigma)\pi^e + O(\pi^{e+1})).$$

As we proved for a above, any $u \in U_1$ can be written as $u = \zeta_p^b u_1$ with $u_1 \equiv 1 \bmod \pi^2$. Each interior term in the binomial expansion of $u_1^p = (1 + O(\pi^2))^p$ other than leading 1 is a multiple of $p\pi^2$ and therefore $O(\pi^{p+1})$; if follows that $u^p = u_1^p \equiv 1 \bmod \pi^{p+1}$. Thus every element of U_1^p is congruent to 1 modulo π^{p+1}, and as you will show on the problem set, the converse holds, that is $U_1^p = \{u \equiv 1 \bmod \pi^{p+1}\}$.

We know from (1) that $\sigma(a)/a^{\omega(\sigma)} \in U_1^p$, so $\sigma(a) = a^{\omega(\sigma)}(1 + O(\pi^{p+1}))$ and therefore

$$\sigma(a) \equiv a^{\omega(\sigma)} \bmod \pi^{p+1}.$$

For $e \leq p$ this is possible only if $\omega(\sigma) = \omega(\sigma)^e$ for every $\sigma \in G$, equivalently, for every $\omega(\sigma) \in \sigma(G) = (\mathbb{Z}/p\mathbb{Z})^\times$, but then $e \equiv 1 \bmod (p-1)$ and we must have $e \geq p$, since $e \geq 2$.

We have shown that every $a \in A$ is represented by an element $\zeta_p^b(1 + c\pi^p + O(\pi^{p+1})) \in U_1$ with $b, c \in \mathbb{Z}$, and therefore lies in the subgroup of U_1/U_1^p generated by ζ_p and $(1 + \pi^p)$, which is an abelian group of exponent p generated by 2 elements, hence isomorphic to a subgroup of $(\mathbb{Z}/p\mathbb{Z})^2$. But this contradicts $A \simeq (\mathbb{Z}/p\mathbb{Z})^3$. □

For $p = 2$ there is an extension of $\mathbb{Q}_2$ with Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, the cyclotomic field $\mathbb{Q}_2(\zeta_{24}) = \mathbb{Q}_2(\zeta_3) \cdot \mathbb{Q}_2(\zeta_8)$. More generally, the unramified cyclotomic field $\mathbb{Q}_2(\zeta_{2^{2^r}-1})$ has Galois group $\mathbb{Z}/2^r\mathbb{Z}$, the totally ramified cyclotomic field $\mathbb{Q}_2(\zeta_{2^{r+2}})$ has Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^r\mathbb{Z}$, and their compositum L has Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2^r\mathbb{Z})^2$. If $K/\mathbb{Q}_2$ is a cyclic extension of degree 2^r that does not lie in L, then one can show that $\mathrm{Gal}(K \cdot L/\mathbb{Q}_2)$ admits a quotient isomorphic to either $(\mathbb{Z}/2\mathbb{Z})^4$, or $(\mathbb{Z}/4\mathbb{Z})^3$, and therefore there exists an extension of $\mathbb{Q}_2$ whose Galois group is isomorphic to one of these two groups. The proof then proceeds by showing that no such extensions exists; we defer the details to the problem set.

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