18.785 Number Theory **Fall 2015**

Problem Set #6

Description

These problems are related to the material covered in Lectures 10-12. Your solutions are to be written up in latex (you can use the latex source for the problem set as a template) and submitted as a pdf-file with a filename of the form `SurnamePset6.pdf` via e-mail to the instructor by 5pm on the date due. Collaboration is permitted/ encouraged, but you must identify your collaborators, and any references you consulted. If there are none, write "**Sources consulted: none**" at the top of your problem set. The first person to spot each non-trivial typo/error in any of the problem sets or lecture notes will receive 1-5 points of extra credit.

Instructions: First do the warm up problems, then pick two of problems 1-3 to solve and write up your answers in latex. Finally, be sure to complete the survey problem 4.

Problem 0.

These are warm up problems that do not need to be turned in.

(a) Prove that the absolute discriminant of a number field must be a square mod 4.

(b) Compute the different ideal of the quadratic extensions $\mathbb{Q}(\sqrt{-2})/\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$.

(c) Determine all the primes that ramify in the cubic fields $\mathbb{Q}[x]/(x^3 - x - 1)$ and $\mathbb{Q}[x]/(x^3 + x + 1)$ and compute their ramification indices.

(d) Let p be an odd prime. Compute the different ideal and absolute discriminant of the cyclotomic extension $\mathbb{Q}(\zeta_p)/\mathbb{Q}$.

Problem 1 The different ideal (50 points)

Let A be a Dedekind domain with fraction field K, let L/K be a finite separable extension, and let B be the integral closure of A in L. Write $L = K(\alpha)$ with $\alpha \in B$ and let $f \in A[x]$ be the minimal polynomial of α, with degree $n = [L : K]$.

(a) By comparing the Laurent series expansion of $1/f(x)$ with it partial fraction decomposition over the splitting field of f (Galois closure of L), prove that

$$\mathrm{T}_{L/K}\left(\frac{\alpha^i}{f'(\alpha)}\right) = \begin{cases} 0 & \text{if } 0 \le i \le n-2; \\ 1 & \text{if } i = n-1; \\ \in A & \text{if } i \ge n. \end{cases}$$

(b) Suppose $B = A[\alpha]$. Prove that $B^* := \{x \in L : T_{L/K}(xb) \in A \text{ for all } b \in B\}$ is the principal fractional B-ideal $(1/f'(\alpha))$. Conclude that $\mathcal{D}_{B/A} = (f'(\alpha))$.

(c) For any $\beta \in B$ with minimal polynomial $g \in A[x]$ define

$$\delta_{B/A}(\beta) = \begin{cases} g'(\beta) & \text{if } L = K(\beta); \\ 0 & \text{otherwise.} \end{cases}$$

One can show that $\mathcal{D}_{B/A}$ is the B-ideal generated by $\{\delta_{B/A}(\beta) : \beta \in B\}$ (you are not required to do this). Prove that if g is the minimal polynomial of $\beta \in B$ for which $L = K(\beta)$ then $N_{B/A}(g'(\beta)) = \pm\mathrm{disc}(g)$. Does this imply that $D_{B/A}$ is generated by the discriminants of all such g? (you may wish to consider Dedekind's cubic field $L = \mathbb{Q}[x]/(x^3 + x^2 - 2x + 8)$ when answering this question).

(d) Let $\mathfrak{c}$ be the conductor of the order $C = A[\alpha]$. Prove that

$$\mathfrak{c} = (B^* : C^*) := \{x \in L : xC^* \subseteq B^*\}.$$

Conclude that if we define $\mathcal{D}_{C/A} := (B : C^*)$ and $D_{C/A} = D(C)$ then we have $\mathcal{D}_{C/A} = \mathfrak{c}\,\mathcal{D}_{B/A}$ and $D_{C/A} = \mathrm{N}_{B/A}(\mathfrak{c})D_{B/A}$, and that $D_{C/A} = \mathrm{N}_{B/A}(\mathcal{D}_{C/A})$.

(e) Let q be a prime of B lying above a prime p of A and suppose the corresponding residue field extension is separable. Prove that $v_{\mathfrak{q}}(\mathcal{D}_{B/A}) \geq e_{\mathfrak{q}} - 1$ with equality if and only if B/A is tamely ramified at $\mathfrak{q}$ (hint: reduce to the case where A is a complete DVR and B/A is totally ramified).

(f) Let p and q be distinct primes congruent to 1 mod 4, let $K := \mathbb{Q}(\sqrt{pq})$, and let $L := \mathbb{Q}(\sqrt{p}, \sqrt{q})$. Prove that $\mathcal{D}_{L/K}$ is the unit ideal; thus L/K is unramified.

Problem 2. Valuation rings (50 points)

An *ordered abelian group* is an abelian group Γ with a total order $\leq$ that is compatible with the group operation. This means that for all $a, b, c \in \Gamma$ the following hold:

$$\begin{array}{rcll} a \leq b \leq a & \implies & a = b & \text{(antisymmetry)} \\ a \leq b \leq c & \implies & a \leq c & \text{(transitivity)} \\ a \not\leq b & \implies & b \leq a & \text{(totality)} \\ a \leq b & \implies & a + c \leq b + c & \text{(compatibility)} \end{array}$$

Note that totality implies reflexivity ($a \leq a$) . Given an ordered abelian group Γ, we define the relations $\geq, <, >$ and the sets $\Gamma_{\leq 0}, \Gamma_{\geq 0}, \Gamma_{<0}$, and $\Gamma_{>0}$ in the obvious way.

A *valuation* v on a field K is a surjective homomorphism $v\colon K^\times \to \Gamma$ to an ordered abelian group Γ that satisfies $v(x + y) \geq \min(v(x), v(y))$ for all $x, y \in K^\times$. The group Γ is called the *value group* of v, and when $\Gamma = \{0\}$ we say that v is the *trivial valuation*. We may extend v to K by defining $v(0) = \infty$, where ∞ is defined to be strictly greater than any element of Γ.

Recall that a *valuation ring* is an integral domain A with fraction field K such that for all $x \in K^\times$ either $x \in A$ or $x^{-1} \in A$ (possibly both).

(a) Let A be a valuation ring with fraction field K, and let $v\colon K^\times \to K^\times/A^\times = \Gamma$ be the quotient map. Show that the relation $\leq$ on Γ defined by

$$v(x) \leq v(y) \iff y/x \in A,$$

makes Γ an ordered abelian group and that v is a valuation on K.

(b) Let K be a field with a non-trivial valuation $v\colon K^\times \to \Gamma$. Prove that the set

$$A := \{x \in K : v(x) \geq 0\}$$

is a valuation ring with fraction field K and that $v(x) \leq v(y) \iff y/x \in A$.

(**c**) Let Γ be an ordered abelian group and let k be a field. For each $a \in \Gamma_{\geq 0}$, let x^a be a formal symbol, and define multiplication of these symbols via $x^a x^b := x^{a+b}$. Let A be the k-algebra whose elements are formal sums $\sum_{a \in I} c_a x^a$, where $c_a \in k$ and the index set $I \subseteq \Gamma_{\geq 0}$ is *well-ordered* (every subset has a minimal element). Let K be the fraction field of A and define $v\colon K^\times \to \Gamma$ by

$$v\left(\frac{\sum c_a x^a}{\sum d_a x^a}\right) = \min\{a : c_a \neq 0\} - \min\{a : d_a \neq 0\}.$$

Prove that v is a valuation on K with value group Γ and valuation ring A.

(**d**) Let $v\colon K^\times \to \Gamma_v$ and $w\colon K^\times \to \Gamma_w$ be two valuations on a field K, and let A_v and A_w be the corresponding valuation rings. Prove that $A_v = A_w$ if and only if there is an order preserving isomorphism $\rho\colon \Gamma_v \to \Gamma_w$ for which $\rho \circ v = w$, in which case we say that v and w are *equivalent*. Thus there is a 1-to-1 correspondence between valuation rings with fraction field K and equivalence classes of valuations on K

(**e**) Let A be an integral domain properly contained in its fraction field K, and let $\mathcal{R}$ be the set of local rings that contain A and are properly contained in K. Partially order $\mathcal{R}$ by writing $R_1 \leq R_2$ if $R_1 \subseteq R_2$ and the maximal ideal of R_1 is contained in the maximal ideal of R_2 (this is known as the *dominance ordering*). Prove that $\mathcal{R}$ contains a maximal element R and that every such R is a valuation ring.

(**f**) Prove that every valuation ring is local and integrally closed, and that the intersection of all valuation rings that contain an integral domain A and lie in its fraction field is equal to the integral closure of A.

(**g**) Prove that a valuation ring that is not a field is a discrete valuation ring if and only if it is noetherian.

Problem 3. Minkowski's lemma and sums of four squares (50 points)

Recall Minkowski's lemma (for $\mathbb{Z}^n$): *if $S \subseteq \mathbb{R}^n$ is a symmetric convex set of volume $\mu(S) > 2^n$ then S contains a nonzero element of $\mathbb{Z}^n$.*

Here *symmetric* means that S is closed under negation, and *convex* means that for all $x, y \in S$ the set $\{tx + (1-t)y : t \in [0,1]\}$ lies in S).

(**a**) Prove that for any measurable $S \subseteq \mathbb{R}^n$ with measure $\mu(S) > 1$ there exist distinct $s, t \in S$ such that $s - t \in \mathbb{Z}^n$, then use this to prove Minkowski's lemma.

(**b**) Prove that Minkowski's lemma is tight in the following sense: show that is is false if either of the words "symmetric" or "convex" is removed, or if the strict inequality $\mu(S) > 2^n$ is weakened to $\mu(S) \geq 2^n$ (give three explicit counter examples).

(**c**) Prove that one can weaken the inequality $\mu(S) > 2^n$ in Minkowski's lemma to $\mu(S) \geq 2^n$ if S is assumed to be compact.

You will now use Minkowski's lemma to prove a theorem of Lagrange, which states that every positive integers is a sum of four integer squares. Let p be an odd prime.

(**d**) Show that $x^2 + y^2 = a$ has a solution (m, n) in $\mathbb{F}_p^2$ for every $a \in \mathbb{F}_p$.

(**e**) Let V be the $\mathbb{F}_p$-span of $\{(m, n, 1, 0), (-n, m, 0, 1)\}$ in $\mathbb{F}_p^4$, where $m^2 + n^2 = -1$. Prove that V is *isotropic*, meaning that $v_1^2 + v_2^2 + v_3^2 + v_4^2 = 0$ for all $v \in V$.

(**f**) Use Minkowski's lemma to prove that p is a sum of four squares.

(**g**) Prove that the set of positive integers that are sums of four squares is closed under multiplication.

(**h**) Prove that every positive integer is the sum of four squares.

Problem 4. Survey

Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast", 5="just right") and the novelty of the material to you (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
10/22	Haar measure, product formula				
10/27	Minkowski bound, finiteness results				

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.

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