18.782 Introduction to Arithmetic Geometry **Fall 2013**
Problem Set #8 **Due: 11/13/2013**

These problems are related to the material covered in Lectures 16-17. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due by the start of class on 11/12/2013 and should be submitted electronically as a pdf-file e-mailed to the instructor. You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions and to identify collaborators and any sources not listed in the syllabus.

Problem 1. Products of varieties and completeness (50 points)

For topological spaces X and Y we write $X \times_{\rm top} Y$ to denote their product as a topological space; if π_X and π_Y are the projection maps, the sets of the form $\pi_X^{-1}(S)$ and $\pi_Y^{-1}(S)$ with S closed generate the closed sets in $X \times_{\rm top} Y$ (under intersection and finite unions). For varieties X and Y we write $X \times Y$ to denote the product variety with the Zariski topology, as defined in Lecture 16. In this problem k denotes an algebraically closed field.

1. Prove that every closed set in $\mathbb{A}^m \times_{\rm top} \mathbb{A}^n$ is closed in $\mathbb{A}^m \times \mathbb{A}^n$, but that the converse is not true for any m and n.

2. Prove that $\mathbb{A}^m \times \mathbb{A}^n$ is isomorphic to $\mathbb{A}^{m+n}$ (for any m and n), but that $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$, nor to any subsvariety of $\mathbb{P}^2$.

3. Define the map $\phi\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ by
$$\phi((x_0 : x_1), (y_0 : y_1)) = [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1].$$
Prove that ϕ is a morphism whose image V is a variety (specify $I(V)$ explicitly). Then prove that $\mathbb{P}^1 \times \mathbb{P}^1 \simeq V$ by giving an inverse morphism.

4. Let $x_0, \ldots, x_m$ and $y_0, \ldots, y_n$ be homogeneous coordinates for $\mathbb{P}^m$ and $\mathbb{P}^n$ respectively, and for $0 \le i \le m$ and $0 \le j \le n$ let z_{ij} be homogeneous coordinates for $\mathbb{P}^N$, where $N = (m + 1)(n + 1) - 1$. Prove that the *Segre morphism* from $\varphi\colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ defined by $z_{ij} = x_iy_j$ is an isomorphism to a projective variety $V \subseteq \mathbb{P}^N$.

5. The field $\mathbb{C}$ is an affine $\mathbb{R}$-algebra (an integral domain finitely generated over $\mathbb{R}$). Show that the tensor product of $\mathbb{R}$-algebras $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not an integral domain, hence not an affine $\mathbb{R}$-algebra. Thus Lemma 16.4, which states that a tensor product of affine algebras is again an affine algebra, depends on the assumption that k is algebraically closed. Explain exactly where in the proof this assumption is used.

6. Let X be a variety. Chevalley's criterion states that if for every variety $Z \subseteq X$ and every valuation ring R of $k(Z)/k$ there is a point $P \in Z$ such that $\mathcal{O}_{P,Z} \subseteq R$, then X is complete. We proved in Lecture 16 that affine varieties of positive dimension are not complete, hence such varieties cannot satisfy Chevalley's criterion. Write down an explicit affine plane curve X and exhibit Z and R for which X does not satisfy Chevalley's criterion. Then indicate which point or points in the projective closure of X address this deficiency.

Problem 2. Valuation rings (50 points)

An *ordered abelian group* is an abelian group Γ with a total order $\leq$ that is compatible with the group operation. This means that for all $a, b, c \in \Gamma$ the following hold:

$$\begin{array}{cccc} a \leq b \leq a & \Longrightarrow & a = b & \text{(antisymmetry)} \\ a \leq b \leq c & \Longrightarrow & a \leq c & \text{(transitivity)} \\ a \not\leq b & \Longrightarrow & b \leq a & \text{(totality)} \\ a \leq b & \Longrightarrow & a + c \leq b + c & \text{(compatibility)} \end{array}$$

Note that totality implies reflexivity ($a \leq a$) . Given an ordered abelian group Γ, we define the relations $\geq, <, >$ and the sets $\Gamma_{\leq 0}, \Gamma_{\geq 0}, \Gamma_{<0}$, and $\Gamma_{>0}$ in the obvious way.

A *valuation* v on a field K is a surjective homomorphism $v\colon K^\times \to \Gamma$ to an ordered abelian group Γ that satisfies $v(x + y) \geq \min(v(x), v(y))$ for all $x, y \in K^\times$. The group Γ is called the *value group* of v, and when $\Gamma = \{0\}$ we say that v is the *trivial valuation*.

1. Let R be a valuation ring with fraction field F, and let $v\colon F^\times \to F^\times/R^\times = \Gamma$ be the quotient map. Show that the relation $\leq$ on Γ defined by

   $$v(x) \leq v(y) \Longleftrightarrow y/x \in R,$$

   makes Γ an ordered abelian group and that v is a valuation on F.

2. Let F be a field and let $v\colon F^\times \to \Gamma$ be a non-trivial valuation. Prove that the set

   $$R_v := v^{-1}(\Gamma_{\geq 0}) \cup \{0\}$$

   is a valuation ring of F and that $v(x) \leq v(y) \Longleftrightarrow y/x \in R_v$ (note that this includes proving that R_v is actually a ring).

3. Let Γ an ordered abelian group, let k be a field, and let A be the k-algebra generated by the set of formal symbols $\{x^a \colon a \in \Gamma_{\geq 0}\}$, with multiplication defined by $x^a x^b = x^{a+b}$. This means that A consists of all sums of the form

   $$\sum_{a \in \Gamma_{\geq 0}} c_a x^a$$

   with $c_a \in k$ and only finitely many nonzero. Let F be the fraction field of A and define the function $v\colon F^\times \to \Gamma$ by

   $$v\left(\frac{\sum c_a x^a}{\sum d_a x^a}\right) = \min\{a : c_a \neq 0\} - \min\{a : d_a \neq 0\}.$$

   Prove that v is a valuation of F with value group Γ.

4. Let $v\colon F^\times \to \Gamma_v$ and $w\colon F^\times \to \Gamma_w$ be two valuations on a field F, and let R_v and R_w be the corresponding valuation rings. Prove that $R_v = R_w$ if and only if there is an order preserving isomorphism $\rho\colon \Gamma_v \to \Gamma_w$ for which $\rho \circ v = w$ (in which case we say that v and w are equivalent). Thus there is a one-to-one correspondence between valuation rings and equivalence classes of valuations.

5. Let D be an integral domain that is properly contained in its fraction field F, and let $\mathcal{R}$ be the set of local rings that contain D and are properly contained in F. Partially order $\mathcal{R}$ by writing $R_1 \le R_2$ if $R_1 \subseteq R_2$ and the maximal ideal of R_1 is contained in the maximal ideal of R_2.[1] Zorn's lemma states that any partially ordered set in which every chain has an upper bound contains at least one maximal element. Use this to prove that $\mathcal{R}$ contains a maximal element R, and then show that any such R is a valuation ring (note: the hypothesis of Zorn's lemma is understood to include the empty chain, so you must prove that $\mathcal{R}$ is nonempty).

6. Prove that every valuation ring R is *integrally closed*. This means that every element of the fraction field of R that is the root of a monic polynomial in $R[x]$ lies in R.

Problem 4. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast"), and the novelty of the material (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
10/31	Products of varieties, Chevalley's criterion				
11/5	Complete varieties, tangent spaces				

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.

[1] This is known as the *dominance* ordering.