Lecture 16

Generating Functions

Recurrences - kth order: $u_{n+k} + a_1 u_{n+k-1} + \ldots a_k u_k = 0$ where $a_1 \ldots a_k$ are given constants, $u_0 \ldots u_{k-1}$ are starting conditions. (Simple case: $u_{n_2} + a u_{n+1} + b u_n = 0$.)

How to solve explicitly - first, write characteristic polynomial (eg., $T^2 + aT + b$) then compute roots (λ and μ, assume $\lambda \neq \mu$)

Then there are some constants α, β such that $u_n = \alpha\lambda^n + \beta\mu^n$, determined with starting conditions. For example,

$$\begin{pmatrix} 1 & 1 \\ \lambda & \mu \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

Why is this the solution? We'll see that any sequence of form $\alpha\lambda^n + \beta\mu^n$ satisfies. Since starting conditions determine the entire sequence, the system of equations above shows that with these values of α, β we have the unique solution.

$$\begin{aligned} u'_n &= \alpha\lambda^n + \beta\mu^n \text{ satisfied the recursion} \\ u_0 &= u'_0, u_1 = u'_1 \\ u_2 &= -au_1 - bu_{-0} \\ &= -au'_1 - bu'_0 \\ &= u'_2, \text{ etc.} \end{aligned}$$

so $u_n = u'_n \; \forall \, n$.

So all we need is to show that $\alpha\lambda^n + \beta\mu^n$ satisfies linear recurrence.

$$u_n + au_{n-1} + bu_{n-2} = 0$$

If $\{u_n\}$ satisfies recurrence, then for any constant α $\{\alpha u_n\}$ also satisfies recurrence, if $\{v_n\}$ also satisfies, then $\{u_n + v_n\}$ also does $\Rightarrow$ solutions to recurrence are closed under taking linear combinations, so it's enough to show that if λ is root of $T^2 + aT + b = 0$, then $\{\lambda^n\}$ satisfies the linear recurrence (and by symmetry μ^n, and linear combination $\alpha\lambda^n + \beta\mu^n$ also does). Plug in to get

$$\lambda^n + a\lambda^{n-1} + b\lambda^{n-2} = 0 \text{ which is by definition true}$$

A more illumination reason - suppose $\{u_n\}$ satisfies recurrence $u_n + au_{n-1} +$

$bu_{n-2} = 0$. Look at generating function with some variable z:

$$
\begin{aligned}
U &= \sum_{n=0}^{\infty} u_n z^n &&= u_0 + u_1 z + \sum_{n\geq 2} u_n z^n \\
azU &= \sum_{n=0}^{\infty} a u_n z^{n+1} &&= \quad a u_0 z + \sum_{n\geq 2} a u_{n-1} z^n \\
bz^2 U &= \sum_{n=0}^{\infty} b u_n z^{n+2} &&= \qquad \sum_{n\geq 2} b u_{n-2} z^n
\end{aligned}
$$

Add to get

$$
\begin{aligned}
U + azU + bz^2U &= u_0 + (u_1 + au_0)z + \sum_{n\geq 2} \underbrace{(u_n + au_{n-1} + bu_{n-2})}_{\text{0 by recurrence definition}} z^n \\
U(1 + az + bz^2) &= u_0 + (u_1 + au_0)z \\
U &= \frac{u_0 + (u_1 + au_0)z}{1 + az + bz^2} \\
&= \frac{\text{linear}(z)}{(1 - \lambda z)(1 - \mu z)} \\
&= \frac{\alpha}{1 - \lambda z} + \frac{\beta}{1 - \mu z} \quad \text{(partial fractions)} \\
\sum_{n=0}^{\infty} u_n z^n &= \alpha(1 + \lambda z + \lambda^2 z^2 \dots) + \beta(1 + \mu z + \mu^2 z^2 \dots) \\
u_n &= (\alpha\lambda^n + \beta\mu^n) z^n
\end{aligned}
$$

Eg. Fibonacci

$$
\begin{aligned}
F_n &= F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1 \\
T^2 &= T + 1 \\
T^2 - T - 1 &= 0 \\
T &= \frac{1 \pm \sqrt{5}}{2}, \quad \lambda = \frac{1 + \sqrt{5}}{2}, \quad \mu = \frac{1 - \sqrt{5}}{2} \\
F_0 &= \alpha + \beta = 0 \\
F_1 &= \alpha\lambda + \beta\mu = 1 \\
\alpha &= \frac{1}{\sqrt{5}} \\
\beta &= -\frac{1}{\sqrt{5}} \\
F_n &= \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right)
\end{aligned}
$$

Eg.

$$
\begin{aligned}
u_n &= 6u_{n-1} - 11u_{n-2} + 6u_{n-3} \\
\text{characteristic polynomial} &= T^3 - 6T^2 + 11T - 6T \\
&= (T-1)(T-2)(T-3) \\
\text{general solution} &= \alpha + \beta 2^n + \gamma 3^n
\end{aligned}
$$

If 2nd order with repeated root $\lambda = \mu$, then general solution $u_n = \alpha\lambda^n + \beta n \lambda^n$.

Eg.

$$
\begin{aligned}
u_n &= 3u_{n-1} - 3u_{n-2} + u_{n-3} \\
&\Rightarrow (T-1)^3 \\
\text{general solution} &= \alpha + \beta n + \gamma n^2
\end{aligned}
$$

Above examples are all homogenous. An example of a non-homogenous linear recurrence:

$$u_n - 5_{n-1} + 6u_{n-2} = n^2$$

Idea - first solve homogenous and find general solution, then find particular solution and add the two solutions.

homogenous: $u_{n_g} = \alpha 2^n + \beta 3^n$.

particular: try similar form $u_n = an^2 + bn + c$, plug into equation

$$u_n - 5u_{n-1} + 6u_{n-2} = 2an^2 + (-14a + 2b)n + (19a - 7b + 2c) = n^2$$
$$a = \frac{1}{2}, b = \frac{7}{2}, c = \frac{15}{2}.$$

general: $u_n = \alpha 2^n + \beta 3^n + \frac{1}{2}(n^2 + 7n + 15)$.

Eg. Look-and-say sequence

$$\begin{aligned}
&1\\
&11\\
&21\\
&1211\\
&111221\\
&312211\\
&13112221\\
&1113213211\\
&\ldots
\end{aligned}$$

(http://en.wikipedia.org/wiki/Look-and-say_sequence) Only 1, 2, 3 appear, lengths satisfy recurrence of degree 71. In any such sequence, eventually splits into sequence of "atomic" elements which never again interact with their neighbors. 92 of these containing only $1, 2, 3$

Some connections of recurrences to number theory.

Eg.

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

Can prove by induction, or explicitly

$$F_n = \frac{\lambda^n - \mu^n}{\lambda - \mu}, \quad \lambda = \frac{1+\sqrt{5}}{2}, \quad \mu = \frac{1-\sqrt{5}}{2}, \quad \lambda + \mu = 1, \quad \lambda\mu = -1$$

$$\begin{aligned}
F_{n+1}F_{n-1} - F_n^2 &= \left(\frac{\lambda^{n+1} - \mu^{n+1}}{\lambda - \mu}\right)\left(\frac{\lambda^{n-1} - \mu^{n-1}}{\lambda - \mu}\right) - \left(\frac{\lambda^n - \mu^n}{\lambda - \mu}\right)^2\\
&= \frac{\lambda^{2n} + \mu^{2n} - \lambda^{n+1}\mu^{n-1} - \lambda^{n-1}\mu^{n+1} - (\lambda^n - \mu^n)^2}{(\lambda - \mu)^2}\\
&= \frac{-\lambda^{n-1}\mu^{n-1}(\lambda^2 - 2\lambda\mu + \mu^2)}{(\lambda - \mu)^2}\\
&= -(\lambda\mu)^{n-1}\\
&= (-1)^n
\end{aligned}$$

In particular,

$$F_{2k+1}F_{2k-1} = F_{2k}^2 + 1$$

Eg. If prime $p > 5$, then $p|F_{p-1}$ or $p|F_{p+1}$ but not both. First compute F_p

$$F_p = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^p - \left(\frac{1-\sqrt{5}}{2}\right)^p\right)$$

$$\text{Now } \left(\frac{1+\sqrt{5}}{2}\right)^p = \frac{1}{2^p}\left(1 + \binom{p}{1}\sqrt{5} + \binom{p}{2}\sqrt{5}^2 + \cdots + \sqrt{5}^p\right)$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^p = \frac{1}{2^p}\left(1 - \binom{p}{1}\sqrt{5} + \binom{p}{2}\sqrt{5}^2 - \cdots - \sqrt{5}^p\right)$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^p - \left(\frac{1-\sqrt{5}}{2}\right)^p = \frac{2}{2^p}\left(\binom{p}{1}\sqrt{5} + \binom{p}{3}\sqrt{5}^3 + \cdots + \sqrt{5}^p\right)$$

So we have

$$F_p = \frac{1}{2^{p-1}}\left(\binom{p}{1} + \binom{p}{3}\sqrt{5}^2 + \cdots + \binom{p}{p-2}\sqrt{5}^{p-3} + \sqrt{5}^{p-1}\right).$$

Want to understand $F_p \mod p$

$$\text{denominator } 2^{p-1} \equiv 1 \mod p \text{ by Fermat's Little Theorem}$$

$$\begin{aligned}\text{numerator} &\equiv \sqrt{5}^{p-1} \mod p \\ &\equiv 5^{\frac{p-1}{2}} \mod p \\ &\equiv \pm 1 \mod p\end{aligned}$$

So $F_p \equiv \pm 1 \mod p$. Then

$$\begin{aligned}F_{p-1}F_{p+1} &= F_p^2 - 1 \mod p \\ F_{p-1}F_{p+1} &\equiv 0 \mod p\end{aligned}$$

Can't divide both because $F_p = F_{p+1} - F_{p-1}$ is not $\equiv 0 \mod p$.

Eg. Sometimes work backwards to show number theoretic properties.

$$\lfloor (1+\sqrt{3})^{2n+1} \rfloor \text{ is divisible by } 2^{n+1}\ \forall\, n$$

Note that $1+\sqrt{3}$ and $1-\sqrt{3}$ are roots of $T^2 - 2T - 2$. Easy to check that $a_n = (1+\sqrt{3})^n + (1-\sqrt{3})^n$ satisfies $a_{n+2} - 2a_{n+1} - 2a_n = 0$ and is an integer sequences. If n odd, then $(1-\sqrt{3})^n$ is negative and between $0, 1$ in absolute value, so $a_n = \lfloor (1+\sqrt{3})^n \rfloor$ for odd n. Set $a_0 = 2, a_1 = 2$, now easy to show that a_{2n} or a_{2n+1} divisible by 2^{n+1} by induction, $a_{2n+2} = 2(a_{2n+1} + a_{2n})$.