18.781 Problem Set 9

Friday, May 11.

Collaboration is allowed and encouraged. However, your writeups should be your own, and you must note on the front the names of the students you worked with.

Extensions will only be given for extenuating circumstances.

In problems 1 through 4, let x be an irrational number, whose continued fraction is $[a_0, a_1, \dots]$. Let p_n/q_n be the n'th convergent to x: its continued fraction is $[a_0, \dots, a_n]$. Also, recall that the digits are obtained as follows: we set $x_0 = x$, and inductively define $a_i = [x_i]$ and $x_{i+1} = 1/(x_i - a_i)$. The continued fraction for x_i is $[a_i, a_{i+1}, \dots]$. In each of these problems, you may use the result of the previous ones as a black box.

1. Recall that
$$x = [a_0, \dots, a_{n-1}, x_n] = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}}.$$
Show that
$$x - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1}(x_n q_{n-1} + q_{n-2})}.$$
Use this to show that the "convergents" p_n/q_n indeed converge to x, and even and odd-numbered convergents lie on opposide sides of x.

   [Note: this fact was swept under the rug in lecture: we showed that the even-numbered convergents increase and the odd-numbered convergents decrease, and that consecutive convergents get closer together; therefore everything converges to some real number, but we never showed that this real number is x (!)]

2. It follows from the above problem that $|x - p_n/q_n| < 1/(q_n q_{n+1})$, and that $|xq_n - p_n| < 1/q_{n+1}$. Show that $|xq_n - p_n| > 1/q_{n+2}$ and therefore that
$$\left| x - \frac{p_{n+1}}{q_{n+1}} \right| < \left| x - \frac{p_n}{q_n} \right|.$$
[Hint: use $x_{n+1} < a_{n+1} + 1$.]

3. (a) Let $n \geq 1$. Show that if a/b is a rational number, with a, b integers and b positive, such that $|bx - a| < |q_n x - p_n|$, then $b \geq q_{n+1}$. [Hint: show that you can write the vector (a, b) as an integer linear combination of (p_n, q_n) and (p_{n+1}, q_{n+1}). Show that the coefficients have opposite signs. Plug these in to $xb - a$ and use that $xq_n - p_n$ and $xq_{n+1} - p_{n+1}$ have opposite signs.]

   (b) Check that (a) implies that $|x - a/b| \geq |x - p_n/q_n|$ for every $1 \leq b \leq q_n$, i.e. p_n/q_n is a best approximation to x among rational numbers with denominators less than or equal to q_n.

4. If a/b is a rational approximation to x as above (a, b integers, b positive), such that
$$\left| x - \frac{a}{b} \right| < \frac{1}{2b^2}$$

then show that a/b must be a convergent of the simple continued fraction of x. [Hint: Choose n such that $q_n \leq b < q_{n+1}$ and use the previous problem.]

5. Let $\phi = (1+\sqrt{5})/2$ be the golden ratio, and let $\kappa > \sqrt{5}$. Show that there are only finitely many rational numbers p/q such that

$$\left|\phi - \frac{p}{q}\right| < \frac{1}{\kappa q^2}$$

(i.e. formalize the heuristic argument from class). You may use the result of problem 4.

6. Let p be a prime congruent to 1 (mod 4), and suppose u an integer such that $u^2 \equiv -1 \pmod{p}$.

   (a) Write the rational number $u/p = [a_0, \ldots, a_n]$, and let i be the largest integer such that $q_i \leq \sqrt{p}$ (here p_i/q_i are the convergents to u/p). Show that

   $$\left|\frac{p_i}{q_i} - \frac{u}{p}\right| < \frac{1}{q_i\sqrt{p}}$$

   and therefore that $|p_i p - u q_i| < \sqrt{p}$.

   (b) Letting $x = q_i$ and $y = p_i p - u q_i$, show that $0 < x^2 + y^2 < 2p$, and that $x^2 + y^2 \equiv 0 \pmod{p}$. Conclude that $p = x^2 + y^2$.

7. Let d be a positive non-square integer. For which positive integers c does the quadratic irrational $([\sqrt{d}] + \sqrt{d})/c$ have a purely periodic expansion?

8. This problem demonstrates an approximation property, without the use of continued fractions (i.e. you should not use convergents here). Let x be an irrational real number.

   (a) Given any positive integer N, show that there is a rational number p/q with p, q integers, and $1 \leq q \leq N$, such that
   $$\left|x - \frac{p}{q}\right| \leq \frac{1}{q(N+1)}.$$
   [Hint: consider the fractional parts of the irrational numbers ix, as i ranges from 0 through N.]

   (b) Use part (a) to show that there are infinitely many rational numbers p/q such that $|x - p/q| < 1/q^2$.

9. Let m be positive integer, and let x have continued fraction $[m, m, m, \ldots]$.

   (a) Compute the value of x.

   (b) Let p_n/q_n be the n'th convergent to x. Write down and solve a linear recurrence with constant coefficients for p_n and for q_n, and thereby calculate an explicit formula for p_n/q_n.

10. (Bonus) Recall the AM-GM inequality:

$$\frac{r_1 + r_2 + \cdots + r_n}{n} \geq \sqrt[n]{r_1 \ldots r_n}$$

for positive real numbers $r_1, \ldots, r_n$. We proved it for $n = 2$.

   (a) Prove the inequality for $n = 2^k$ any power of 2. [Hint: proceed by induction, grouping the terms into two halves]

   (b) Prove the inequality for any n, by choosing a k such that $2^{k-1} < n \leq 2^k$, and applying the inequality from part (a) to the 2^k numbers $r_1, \ldots, r_n, r, r, \ldots, r$, where r is chosen appropriately.

11. (Bonus) Given any constant c, show that there exists an irrational number x and infinitely many rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{k^c}.$$