8 General Lie Groups

In §6 we considered semisimple Lie groups; now we shall make a few remarks about another major class of Lie groups, the solvable Lie groups.

If $\mathfrak{g}$ is a Lie algebra the vector space spanned by all elements $[X, Y]$ $(X, Y \in \mathfrak{g})$ is an ideal in $\mathfrak{g}$, called the *derived algebra* $\mathcal{D}\mathfrak{g}$ of $\mathfrak{g}$. Then $\mathcal{D}^n\mathfrak{g}$ is defined inductively by $\mathcal{D}^0\mathfrak{g} = \mathfrak{g}$, $\mathcal{D}^n\mathfrak{g} = \mathcal{D}(\mathcal{D}^{n-1}\mathfrak{g})$. The Lie algebra is *solvable* if $\mathcal{D}^n\mathfrak{g} = 0$ for some $n \geq 0$. A Lie group is solvable if its Lie algebra is solvable.

Proposition 8.1. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a}$ or $\mathfrak{b}$ solvable ideals. Then $\mathfrak{a}+\mathfrak{b}$ is also a solvable ideal.*

Proof: $\mathfrak{a} + \mathfrak{b}$ is of course an ideal. Now consider the map $A + B \to B(\mod (\mathfrak{a} \cap \mathfrak{b}))$ which is a well-defined homomorphism with kernel $\mathfrak{a}$. Thus $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \sim \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$ so $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a}$ is solvable. Image of $\mathcal{D}^k(\mathfrak{a}+\mathfrak{b})$ in $(\mathfrak{a}+\mathfrak{b}/\mathfrak{a})$ is contained in $\mathcal{D}^k((\mathfrak{a}+\mathfrak{b})/\mathfrak{a})$ so $\mathcal{D}^k(\mathfrak{a}+\mathfrak{b}) \subset \mathfrak{a}$ for k large. Since $\mathfrak{a}$ is solvable, $\mathcal{D}^\ell(\mathfrak{a} + \mathfrak{b}) = 0$ for ℓ large so $\mathfrak{a} + \mathfrak{b}$ is solvable.

Thus if $\mathfrak{r} \subset \mathfrak{g}$ is a solvable ideal of maximal dimension then every solvable ideal is contained in $\mathfrak{r}$. Thus $\mathfrak{r}$ is the union of all solvable ideals. It is called the *radical.* The following basic result we now state without proof.

Theorem 8.2 (Levi decomposition.). *Each Lie algebra $\mathfrak{g}$ has the decomposition*

$$\mathfrak{g} = \mathfrak{r} + \mathfrak{s} \qquad \mathfrak{r} \cap \mathfrak{s} = 0$$

where $\mathfrak{r}$ is the radical and $\mathfrak{s}$ is semi-simple.

Using that theorem we can prove the following fundamental theorem.

Theorem 8.3. *For each Lie algebra $\mathfrak{g}$ over $\mathbf{R}$ there exists a Lie group G with Lie algebra $\mathfrak{g}$.*

The local version of this theorem is the so-called third theorem of Lie which will be proved later, The global version was proved by Cartan in 1930. It is actually a simple consequence of Theorem 8.2.

If A and B are abstract groups and $b \to \sigma_b$ a homomorphism of B into $\mathrm{Aut}\,(A)$ the semi-direct product $A \times_\sigma B$ is the group defined by the product

$$(a, b)(a', b') = (a\sigma_b(a'), bb')$$

on $A \times B$. This is indeed a group containing A as a normal subgroup.

Proposition 8.4 *Suppose A and B are connected Lie groups, σ an analytic homomorphism of B into* $\mathrm{Aut}\,(A)$. *Let $\mathfrak{a}$ and $\mathfrak{b}$ denote their respective Lie algebras. Then the group $G = A \times_\sigma B$ has Lie algebra*

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$$

with the bracket relation

$$[X + Y, X' + Y'] = [X, X'] + d\psi(Y)(X') - d\psi(Y')(X) + [Y, Y'],$$

where $X, X' \in \mathfrak{a}$, $Y, Y' \in \mathfrak{b}$ and ψ is the map $b \to d\sigma_b$ of B into $\mathrm{Aut}\,(\mathfrak{a})$.

Proof Since $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{g}$ it remains to prove

$$[X, Y] = -d\psi(Y)(X), \quad X \in \mathfrak{a}, Y \in \mathfrak{b}.$$

The differential $d\psi$ is a homomorphism of $\mathfrak{b}$ into $\partial(\mathfrak{a})$, the Lie algebra of derivations of $\mathfrak{a}$. We have

$$d\sigma_{\exp tY} = \psi(\exp tY) = e^{t\, d\psi(Y)}.$$

Hence by the multiplication in $A \times_\sigma B$,

$$\begin{aligned}\exp(-tX)\exp(-tY)\exp(tX)\exp(tY) &= \exp(-tX)\sigma_{\exp(-tY)}(\exp(tX)) \\ = \exp(-tX)\exp(t\, d\sigma_{\exp(-tY)}(X)) &= \exp(-tX)\exp(te^{-t\, d\psi(Y)}).\end{aligned}$$

Expanding this in powers of t we deduce from Lemma 1.8, $[X, Y] = -d\psi(Y)(X)$ as desired.

Lemma 8.5 *If $\mathfrak{g}$ is a solvable Lie algebra then there exists a Lie group G with Lie algebra $\mathfrak{g}$.*

Proof. This is proved by induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1$ we take $G = \mathbf{R}$. If $\dim \mathfrak{g} > 1$ then $\mathfrak{D}\mathfrak{g} \neq \mathfrak{g}$ so there exists a subspace $\mathfrak{h}$ such that $\mathfrak{D}\mathfrak{g} \subset \mathfrak{h}$ and $\dim \mathfrak{h} = \dim \mathfrak{g} - 1$. Let $X \in \mathfrak{g}$, $X \notin \mathfrak{h}$. Then $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{D}\mathfrak{g} \subset \mathfrak{h}$ so $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} + \mathbf{R}X$. Let by induction H be a simply connected Lie group with Lie algebra $\mathfrak{h}$ and A a Lie group with Lie algebra $\mathbf{R}X$. The derivation $Y \to [X, Y]$ of $\mathfrak{h}$ extends to a homomorphism $\sigma : A \to \mathrm{Aut}\,(\mathfrak{h})$ so by Proposition 8.4, $H \times_\sigma A$ serves as the desired G.

Now let $\mathfrak{g}$ be an arbitrary Lie algebra over $\mathbf{R}$. Assuming the Levi decomposition $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$, we deduce from the Lemma 8.5, Proposition 8.4 and Corollary 6.5 that $\mathfrak{g}$ is the Lie algebra of a Lie group.

§ 9 Differential Forms*

We shall now deal with the general theory of differential forms on a manifold M and in a later section specialize to the case when M is a Lie group.

Let A be a commutative ring with identity element, E a module over A. Let E^* denote the set of all A-linear mappings of E into A. Then E^* is an A-module in an obvious fashion. It is called the *dual* of E.

Definition. Let M be a C^∞ manifold and put $\mathfrak{F} = C^\infty(M)$. Let $\mathfrak{D}_1(M)$ denote the dual of the $\mathfrak{F}$-module $\mathfrak{D}^1(M)$. The elements of $\mathfrak{D}_1(M)$ are called the *differential 1-forms* on M (or just 1-forms on M).

Let $X \in \mathfrak{D}^1(M)$, $\omega \in \mathfrak{D}_1(M)$. Suppose that X vanishes on an open set V. Then the function $\omega(X)$ vanishes on V. In fact, if $p \in V$, there exists a function $f \in C^\infty(M)$ such that $f = 0$ in a compact neighborhood of p and $f = 1$ outside V. Then $fX = X$ and since ω is $\mathfrak{F}$-linear, $\omega(X) = f\omega(X)$. Hence $(\omega(X))(p) = 0$. This shows also that a 1-form on M induces a 1-form on any open submanifold of M. Using (3) we obtain the following lemma.

Lemma 2.1. *Let $X \in \mathfrak{D}^1(M)$ and $\omega \in \mathfrak{D}_1(M)$. If $X_p = 0$ for some $p \in M$, then the function $\omega(X)$ vanishes at p.*

This lemma shows that given $\omega \in \mathfrak{D}_1(M)$, we can define the linear function ω_p on M_p by putting $\omega_p(X_p) = (\omega(X))(p)$ for $X \in \mathfrak{D}^1(M)$. The set $\mathfrak{D}_1(p) = \{\omega_p : \omega \in \mathfrak{D}_1(M)\}$ is a vector space over $\boldsymbol{R}$.

We have seen that a 1-form on M induces a 1-form on any open submanifold. On the other hand, suppose θ is a 1-form on an open submanifold V of M and p a point in V. Then there exists a 1-form $\bar{\theta}$ on M, and an open neighborhood N of p, $p \in N \subset V$, such that θ and $\bar{\theta}$ induce the same 1-form on N. In fact, let C be a compact neighborhood of p contained in V and let N be the interior of C. Select $\psi \in C^\infty(M)$ of

***Note that in this section we have kept the original numbering**

compact support contained in V such that $\psi = 1$ on C. Then a 1-form $\tilde{\theta}$ with the desired property can be defined by

$$\tilde{\theta}(X) = \psi\theta(X_V) \text{ on } V, \qquad \tilde{\theta}(X) = 0 \text{ outside } V,$$

where $X \in \mathfrak{D}^1(M)$ and X_V denotes the vector field on V induced by X.

Lemma 2.2. *The space $\mathfrak{D}_1(p)$ coincides with M_p^*, the dual of M_p.*

We already know that $\mathfrak{D}_1(p) \subset M_p^*$. Now let $\{x_1, ..., x_m\}$ be a system of coordinates valid on an open neighborhood U of p. Owing to (3), there exist 1-forms ω^i on U such that† $\omega^i(\partial/\partial x_j) = \delta_{ij}$ $(1 \leqslant i, j \leqslant m)$. Let $L \in M_p^*$, $l_i = L((\partial/\partial x_i)_p)$ and $\theta = \sum_{i=1}^m l_i\omega^i$. Then there exists a 1-form $\tilde{\theta}$ on M and an open neighborhood N of p $(N \subset U)$ such that $\tilde{\theta}$ and θ induce the same form on N. Then $(\tilde{\theta})_p = L$ and the lemma is proved.

Each $X \in \mathfrak{D}^1(M)$ induces an $\mathfrak{F}$-linear mapping $\omega \to \omega(X)$ of $\mathfrak{D}_1(M)$ into $\mathfrak{F}$. If $X_1 \neq X_2$, the induced mappings are different (due to Lemma 2.2). Thus, $\mathfrak{D}^1(M)$ can be regarded as a subset of $(\mathfrak{D}_1(M))^*$.

Lemma 2.3. *The module $\mathfrak{D}^1(M)$ coincides with the dual of the module $\mathfrak{D}_1(M)$.*

Proof. Let $F \in \mathfrak{D}_1(M)^*$. Then $F(f\omega) = fF(\omega)$ for all $f \in C^\infty(M)$ and all $\omega \in \mathfrak{D}_1(M)$. This shows that if ω vanishes on an open set V, $F(\omega)$ also vanishes on V. Let $p \in M$ and $\{x_1, ..., x_m\}$ a system of local coordinates valid on an open neighborhood U of p. Each 1-form on U can be written $\sum_{i=1}^m f_i\omega^i$ where $f_i \in C^\infty(U)$ and ω^i has the same meaning as above. It follows easily that $F(\omega)$ vanishes at p whenever $\omega_p = 0$; consequently, the mapping $\omega_p \to (F(\omega))(p)$ is a well-defined linear function on $\mathfrak{D}_1(p)$. By Lemma 2.2 there exists a unique vector $X_p \in M_p$ such that $(F(\omega))(p) = \omega_p(X_p)$ for all $\omega \in \mathfrak{D}_1(M)$. Thus, F gives rise to a family X_p $(p \in M)$ of tangent vectors to M. For each $q \in U$ we can write

$$X_q = \sum_{i=1}^m a_i(q)\left(\frac{\partial}{\partial x_i}\right)_q,$$

where $a_i(q) \in \boldsymbol{R}$. For each i $(1 \leqslant i \leqslant m)$ there exists a 1-form $\tilde{\omega}^i$ on M which coincides with ω^i in an open neighborhood N_p of p, $(N_p \subset U)$. Then $(F(\tilde{\omega}^i))(q) = \tilde{\omega}^i_q(X_q) = a_i(q)$ for $q \in N_p$. This shows that the functions a_i are differentiable. If $f \in C^\infty(M)$ and we denote the function $q \to X_qf$ $(q \in M)$ by Xf, then the mapping $f \to Xf$ is a vector field on M which satisfies $\omega(X) = F(\omega)$ for all $\omega \in \mathfrak{D}_1(M)$. This proves the lemma.

† As usual, $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$.

Let A be a commutative ring with identity element. Let I be a set and suppose that for each $i \in I$ there is given an A-module E_i. The product set $\Pi_{i \in I} E_i$ can be turned into an A-module as follows: If $e = \{e_i\}$, $e' = \{e_i'\}$ are any two elements in ΠE_i (where $e_i, e_i' \in E_i$), and $a \in A$, then $e + e'$ and ae are given by

$$(e + e')_i = e_i + e_i', \quad (ae)_i = ae_i \qquad \text{for } i \in I.$$

The module ΠE_i is called the *direct product* of the modules E_i. The *direct sum* $\Sigma_{i \in I} E_i$ is defined as the submodule of ΠE_i consisting of those elements $e = \{e_i\}$ for which all $e_i = 0$ except for finitely many i.

Suppose now the set I is finite, say $I = \{1, ..., s\}$. A mapping $f: E_1 \times ... \times E_s \to F$ where F is an A-module is said to be *A-multilinear* if it is A-linear in each argument. The set of all A-multilinear mappings of $E_1 \times ... \times E_s$ into F is again an A-module as follows:

$$(f + f')(e_1, ..., e_s) = f(e_1, ..., e_s) + f'(e_1, ..., e_s),$$
$$(af)(e_1, ..., e_s) = a(f(e_1, ..., e_s)).$$

Suppose that all the factors E_i coincide. The A-multilinear mapping f is called *alternate* if $f(X_1, ..., X_s) = 0$ whenever at least two X_i coincide.

Now, let M be a C^∞ manifold and as usual we put $\mathfrak{F} = C^\infty(M)$. If s is an integer, $s \geqslant 1$, we consider the $\mathfrak{F}$-module

$$\mathfrak{D}^1 \times \mathfrak{D}^1 \times ... \times \mathfrak{D}^1 \qquad (s \text{ times})$$

and let $\mathfrak{D}_s$ denote the $\mathfrak{F}$-module of all $\mathfrak{F}$-multilinear mappings of $\mathfrak{D}^1 \times ... \times \mathfrak{D}^1$ into $\mathfrak{F}$. Similarly $\mathfrak{D}^r$ denotes the $\mathfrak{F}$-module of all $\mathfrak{F}$-multilinear mappings of

$$\mathfrak{D}_1 \times \mathfrak{D}_1 \times ... \times \mathfrak{D}_1 \qquad (r \text{ times})$$

into $\mathfrak{F}$. This notation is permissible since we have seen that the modules $\mathfrak{D}^1$ and $\mathfrak{D}_1$ are duals of each other. More generally, let $\mathfrak{D}_s^r$ denote the $\mathfrak{F}$-module of all $\mathfrak{F}$-multilinear mappings of

$$\mathfrak{D}_1 \times ... \times \mathfrak{D}_1 \times \mathfrak{D}^1 \times ... \times \mathfrak{D}^1 \qquad (\mathfrak{D}_1\ r \text{ times},\ \mathfrak{D}^1\ s \text{ times})$$

into $\mathfrak{F}$. We often write $\mathfrak{D}_s^r(M)$ instead of $\mathfrak{D}_s^r$. We have $\mathfrak{D}_0^r = \mathfrak{D}^r$, $\mathfrak{D}_s^0 = \mathfrak{D}_s$ and we put $\mathfrak{D}_0^0 = \mathfrak{F}$.

A *tensor field* T on M of type (r, s) is by definition an element of $\mathfrak{D}_s^r(M)$. This tensor field T is said to be *contravariant* of degree r, *covariant* of degree s. In particular, the tensor fields of type $(0, 0)$, $(1, 0)$, and $(0, 1)$ on M are just the differentiable functions on M, the vector fields on M and the 1-forms on M, respectively.

If p is a point in M, we define $\mathfrak{D}^r_s(p)$ as the set of all $\boldsymbol{R}$-multilinear mappings of

$$M_p^* \times \dots \times M_p^* \times M_p \times \dots \times M_p \qquad (M_p^* \;\; r \text{ times}, M_p \;\; s \text{ times})$$

into $\boldsymbol{R}$. The set $\mathfrak{D}^r_s(p)$ is a vector space over $\boldsymbol{R}$ and is nothing but the tensor product

$$M_p \otimes \dots \otimes M_p \otimes M_p^* \otimes \dots \otimes M_p^* \qquad (M_p \;\; r \text{ times}, M_p^* \;\; s \text{ times})$$

or otherwise written

$$\mathfrak{D}^r_s(p) = \otimes^r M_p \otimes {}^s M_p^*.$$

We also put $\mathfrak{D}^0_0(p) = \boldsymbol{R}$. Consider now an element $T \in \mathfrak{D}^r_s(M)$. We have

$$T(g_1\theta_1, \dots, g_r\theta_r, f_1Z_1, \dots, f_sZ_s) = g_1 \dots g_r f_1 \dots f_s T(\theta_1, \dots, \theta_r, Z_1, \dots, Z_s)$$

for $f_i, g_j \in C^\infty(M)$, $Z_i \in \mathfrak{D}^1(M)$, $\theta_j \in \mathfrak{D}_1(M)$. It follows from Lemma 1.2 that if some θ_j or some Z_i vanishes on an open set V, then the function $T(\theta_1, \dots, \theta_r, Z_1, \dots, Z_s)$ vanishes on V. Let $\{x_1, \dots, x_m\}$ be a system of coordinates valid on an open neighborhood U of p. Then there exist vector fields X_i $(1 \leqslant i \leqslant m)$ and 1-forms ω_j $(1 \leqslant j \leqslant m)$ on M and an open neighborhood N of p, $p \in N \subset U$ such that on N

$$X_i = \frac{\partial}{\partial x_i}, \qquad \omega_j(X_i) = \delta_{ij} \qquad (1 \leqslant i, j \leqslant m).$$

On N, Z_i and θ_j can be written

$$Z_i = \sum_{k=1}^m f_{ik} X_k, \qquad \theta_j = \sum_{l=1}^m g_{jl}\omega_l,$$

where $f_{ik}, g_{jl} \in C^\infty(N)$, and by the remark above we have for $q \in N$,

$$T(\theta_1, \dots, \theta_r, Z_1, \dots, Z_s)(q)$$

$$= \sum_{l_j=1, k_i=1}^m g_{1l_1} \cdots g_{rl_r} f_{1k_1} \cdots f_{sk_s} T(\omega_{l_1}, \dots, \omega_{l_r}, X_{k_1}, \dots, X_{k_s})(q).$$

This shows that $T(\theta_1, \dots, \theta_r, Z_1, \dots, Z_s)(p) = 0$ if some θ_j or some Z_i vanishes at p. We can therefore define an element $T_p \in \mathfrak{D}^r_s(p)$ by the condition

$$T_p((\theta_1)_p, \dots, (\theta_r)_p, (Z_1)_p, \dots, (Z_s)_p) = T(\theta_1, \dots, \theta_r, Z_1, \dots, Z_s)(p).$$

The tensor field T thus gives rise to a family $T_p, p \in M$, where $T_p \in \mathfrak{D}^r_s(p)$. It is clear that if $T_p = 0$ for all p, then $T = 0$. The element $T_p \in \mathfrak{D}^r_s(p)$ depends differentiably on p in the sense that if N is a coordinate neighborhood of p and T_q (for $q \in N$) is expressed as above in terms of bases for $\mathfrak{D}_1(N)$ and $\mathfrak{D}^1(N)$, then the coefficients are differentiable functions on N. On the other hand, if there is a rule $p \to T(p)$ which to each $p \in M$ assigns a member $T(p)$ of $\mathfrak{D}^r_s(p)$ in a differentiable manner (as described above), then there exists a tensor field T of type (r, s) such that $T_p = T(p)$ for all $p \in M$. In the case when M is analytic it is clear how to define analyticity of a tensor field T, generalizing the notion of an analytic vector field.

The vector spaces $\mathfrak{D}^r_s(p)$ and $\mathfrak{D}^s_r(p)$ are dual to each other under the nondegenerate bilinear form $\langle\, ,\, \rangle$ on $\mathfrak{D}^r_s(p) \times \mathfrak{D}^s_r(p)$ defined by the formula

$$\langle e_1 \otimes \dots \otimes e_r \otimes f_1 \otimes \dots \otimes f_s, e_1' \otimes \dots \otimes e_s' \otimes f_1' \otimes \dots \otimes f_r' \rangle = \prod_{i,j} f_j(e_j') f_i'(e_i),$$

where e_i, e_j' are members of a basis of M_p, f_j, f_i' are members of a dual basis of M_p^*. It is then obvious that the formula holds if e_i, e_j' are arbitrary elements of M_p and f_j, f_i' are arbitrary elements of M_p^*. In particular, the form $\langle\, ,\, \rangle$ is independent of the choice of basis used in the definition.

Each $T \in \mathfrak{D}^r_s(M)$ induces an $\mathfrak{F}$-linear mapping of $\mathfrak{D}^s_r(M)$ into $\mathfrak{F}$ given by the formula

$$(T(S))(p) = \langle T_p, S_p \rangle \qquad \text{for } S \in \mathfrak{D}^s_r(M).$$

If $T(S) = 0$ for all $S \in \mathfrak{D}^s_r(M)$, then $T_p = 0$ for all $p \in M$, so $T = 0$. Consequently, $\mathfrak{D}^r_s(M)$ can be regarded as a subset of $(\mathfrak{D}^s_r(M))^*$. We have now the following generalization of Lemma 2.3.

Lemma 2.3′. *The module $\mathfrak{D}^r_s(M)$ is the dual of $\mathfrak{D}^s_r(M)$ $(r, s \geqslant 0)$.*

Except for a change in notation the proof is the same as that of Lemma 2.3. To emphasize the duality we sometimes write $\langle T, S \rangle$ instead of $T(S)$, $(T \in \mathfrak{D}^r_s, S \in \mathfrak{D}^s_r)$.

Let $\mathfrak{D}$ (or $\mathfrak{D}(M)$) denote the direct sum of the $\mathfrak{F}$-modules $\mathfrak{D}^r_s(M)$,

$$\mathfrak{D} = \sum_{r,s=0}^{\infty} \mathfrak{D}^r_s.$$

Similarly, if $p \in M$ we consider the direct sum

$$\mathfrak{D}(p) = \sum_{r,s=0}^{\infty} \mathfrak{D}^r_s(p).$$

The vector space $\mathfrak{D}(p)$ can be turned into an associative algebra over R as follows: Let $a = e_1 \otimes \ldots \otimes e_r \otimes f_1 \otimes \ldots \otimes f_s$, $b = e'_1 \otimes \ldots \otimes e'_\rho \otimes f'_1 \ldots \otimes f'_\sigma$, where e_i, e'_i are members of a basis for M_p, f_j, f'_j are members of a dual basis for M_p^*. Then $a \otimes b$ is defined by the formula

$$a \otimes b = e_1 \otimes \ldots \otimes e_r \otimes e'_1 \otimes \ldots \otimes e'_\rho \otimes f_1 \otimes \ldots \otimes f_s \otimes f'_1 \otimes \ldots \otimes f'_\sigma.$$

We put $a \otimes 1 = a$, $1 \otimes b = b$ and extend the operation $(a, b) \to a \otimes b$ to a bilinear mapping of $\mathfrak{D}(p) \times \mathfrak{D}(p)$ into $\mathfrak{D}(p)$. Then $\mathfrak{D}(p)$ is an associative algebra over R. The formula for $a \otimes b$ now holds for arbitrary elements $e_i, e'_i \in M_p$ and $f_j, f'_j \in M_p^*$. Consequently, the multiplication in $\mathfrak{D}(p)$ is independent of the choice of basis.

The tensor product $\otimes$ in $\mathfrak{D}$ is now defined as the $\mathfrak{F}$-bilinear mapping $(S, T) \to S \otimes T$ of $\mathfrak{D} \times \mathfrak{D}$ into $\mathfrak{D}$ such that

$$(S \otimes T)_p = S_p \otimes T_p, \qquad S \in \mathfrak{D}^r_s,\ T \in \mathfrak{D}^\rho_\sigma,\ p \in M.$$

This turns the $\mathfrak{F}$-module $\mathfrak{D}$ into a ring satisfying

$$f(S \otimes T) = fS \otimes T = S \otimes fT$$

for $f \in \mathfrak{F}$, $S, T \in \mathfrak{D}$. In other words, $\mathfrak{D}$ is an associative algebra over the ring $\mathfrak{F}$. The algebras $\mathfrak{D}$ and $\mathfrak{D}(p)$ are called the ***mixed tensor algebras*** over M and M_p, respectively. The submodules

$$\mathfrak{D}^* = \sum_{r=0}^{\infty} \mathfrak{D}^r, \qquad \mathfrak{D}_* = \sum_{s=0}^{\infty} \mathfrak{D}_s$$

are subalgebras of $\mathfrak{D}$ (also denoted $\mathfrak{D}^*(M)$ and $\mathfrak{D}_*(M)$) and the subspaces

$$\mathfrak{D}^*(p) = \sum_{r=0}^{\infty} \mathfrak{D}^r(p), \qquad \mathfrak{D}_*(p) = \sum_{s=0}^{\infty} \mathfrak{D}_s(p)$$

are subalgebras of $\mathfrak{D}(p)$.

Now let r, s be two integers $\geqslant 1$, and let i, j be integers such that $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$. Consider the R-linear mapping $C^i{}_j$: $\mathfrak{D}^r_s(p) \to \mathfrak{D}^{r-1}_{s-1}(p)$ defined by

$$C^i{}_j(e_1 \otimes \ldots \otimes e_r \otimes f_1 \otimes \ldots \otimes f_s) = \langle e_i, f_j \rangle (e_1 \otimes \ldots \hat{e}_i \ldots \otimes e_r \otimes f_1 \otimes \ldots \hat{f}_j \ldots \otimes f_s),$$

where $e_1, \ldots, e_r$ are members of a basis of M_p, $f_1, \ldots, f_s$ are members of the dual basis of M_p^*. (The symbol ^ over a letter means that the letter is missing.) Now that the existence of $C^i{}_j$ is established, we note that the formula for $C^i{}_j$ holds for arbitrary elements $e_1, ..., e_r \in M_p$, $f_1, ..., f_s \in M_p^*$. In particular, $C^i{}_j$ is independent of the choice of basis.

There exists now a unique $\mathfrak{F}$-linear mapping $C^i{}_j$: $\mathfrak{D}^r_s(M) \to \mathfrak{D}^{r-1}_{s-1}(M)$ such that

$$(C^i{}_j(T))_p = C^i{}_j(T_p)$$

for all $T \in \mathfrak{D}^r_s(M)$ and all $p \in M$. This mapping satisfies the relation

$$C^i{}_j(X_1 \otimes ... \otimes X_r \otimes \omega_1 \otimes ... \otimes \omega_s)$$
$$= \langle X_i, \omega_j \rangle (X_1 \otimes ... \hat{X}_i ... \otimes X_r \otimes \omega_1 \otimes ... \hat{\omega}_j ... \otimes \omega_s)$$

for all $X_1, ..., X_r \in \mathfrak{D}^1$, $\omega_1, ..., \omega_s \in \mathfrak{D}_1$. The mapping $C^i{}_j$ is called the *contraction* of the ith contravariant index and the jth covariant index.

The Grassmann Algebra

As before, M denotes a C^∞ manifold and $\mathfrak{F} = C^\infty(M)$. If s is an integer $\geqslant 1$, let $\mathfrak{A}_s$ (or $\mathfrak{A}_s(M)$) denote the set of alternate $\mathfrak{F}$-multilinear mappings of $\mathfrak{D}^1 \times ... \times \mathfrak{D}^1$ (s times) into $\mathfrak{F}$. Then $\mathfrak{A}_s$ is a submodule of $\mathfrak{D}_s$. We put $\mathfrak{A}_0 = \mathfrak{F}$ and let $\mathfrak{A}$ (or $\mathfrak{A}(M)$) denote the direct sum $\mathfrak{A} = \sum_{s=0}^\infty \mathfrak{A}_s$ of the $\mathfrak{F}$-modules $\mathfrak{A}_s$. The elements of $\mathfrak{A}(M)$ are called *exterior differential forms* on M. The elements of $\mathfrak{A}_s$ are called differential s-forms (or just s-forms).

Let $\mathfrak{S}_s$ denote the group of permutations of the set $\{1, 2, ..., s\}$. Each $\sigma \in \mathfrak{S}_s$ induces an $\mathfrak{F}$-linear mapping of $\mathfrak{D}^1 \times ... \times \mathfrak{D}^1$ onto itself given by

$$(X_1, ..., X_s) \to (X_{\sigma^{-1}(1)}, ..., X_{\sigma^{-1}(s)}) \qquad (X_i \in \mathfrak{D}^1).$$

This mapping will also be denoted by σ. Since each $d \in \mathfrak{D}_s$ is a multilinear map of $\mathfrak{D}^1 \times ... \times \mathfrak{D}^1$ into $\mathfrak{F}$, the mapping $d \circ \sigma^{-1}$ is well defined. Moreover, the mapping $d \to d \circ \sigma^{-1}$ is a one-to-one $\mathfrak{F}$-linear mapping of $\mathfrak{D}_s$ onto itself. If we write $\sigma \cdot d = d \circ \sigma^{-1}$ we have $\sigma\tau \cdot d = \sigma \cdot (\tau \cdot d)$. Let $\epsilon(\sigma) = 1$ or -1 according to whether σ is an even or an odd permutation. Consider the linear transformation A_s: $\mathfrak{D}_s \to \mathfrak{D}_s$ given by

$$A_s(d_s) = \frac{1}{s!} \sum_{\sigma \in \mathfrak{S}_s} \epsilon(\sigma)\, \sigma \cdot d_s, \qquad d_s \in \mathfrak{D}_s.$$

If $s = 0$, we put $A_s(d_s) = d_s$. We extend A_s to an $\mathfrak{F}$-linear mapping A: $\mathfrak{D}_* \to \mathfrak{D}_*$ by putting $A(d) = \sum_{s=0}^\infty A_s(d_s)$ if $d = \sum_{s=0}^\infty d_s$, $d_s \in \mathfrak{D}_s$.

If $\tau \in \mathfrak{S}_s$, we have

$$\tau A_s(d_s) = \frac{1}{s!} \sum_{\sigma \in \mathfrak{S}_s} \epsilon(\sigma)\ \tau \cdot (\sigma \cdot d_s) = \frac{1}{s!} \sum_{\sigma \in \mathfrak{S}_s} \epsilon(\sigma)\,(\tau\sigma) \cdot d_s$$

$$= \epsilon(\tau) \frac{1}{s!} \sum_{\sigma \in \mathfrak{S}_s} \epsilon(\sigma)\, \sigma \cdot d_s.$$

Hence, $\tau \cdot \big(A_s(d_s)\big) = \epsilon(\tau)\, A_s(d_s)$. This shows that $A_s(\mathfrak{D}_s) \subset \mathfrak{A}_s$ and $A(\mathfrak{D}_*) \subset \mathfrak{A}$. On the other hand, if $d_s \in \mathfrak{A}_s$, then $\sigma \cdot d_s = \epsilon(\sigma)\, d_s$ for each $\sigma \in \mathfrak{S}_s$. Since $\epsilon(\sigma)^2 = 1$, we find that

$$A_s(d_s) = d_s \qquad \text{if } d_s \in \mathfrak{A}_s.$$

It follows that $A^2 = A$ and $A(\mathfrak{D}_*) = \mathfrak{A}$; in other words, A is a projection of $\mathfrak{D}_*$ onto $\mathfrak{A}$. The mapping A is called *alternation.*

Let N denote the kernel of A. Obviously N is a submodule of $\mathfrak{D}_*$.

Lemma 2.4. *The module N is a two-sided ideal in $\mathfrak{D}_*$.*

It suffices to show that if $n_r \in N \cap \mathfrak{D}_r$, $d_s \in \mathfrak{D}_s$, then $A_{r+s}(n_r \otimes d_s) = A_{s+r}(d_s \otimes n_r) = 0$. Let $b_{r+s} = A_{r+s}(n_r \otimes d_s)$; then

$$(r+s)!\, b_{r+s} = \sum_{\sigma \in \mathfrak{S}_{r+s}} \epsilon(\sigma)\, \sigma \cdot (n_r \otimes d_s),$$

where

$$\sigma \cdot (n_r \otimes d_s)(X_1, \ldots, X_{r+s}) = n_r(X_{\sigma(1)}, \ldots, X_{\sigma(r)})\, d_s(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)}).$$

The elements in $\mathfrak{S}_{r+s}$ which leave each number $r+1, \ldots, r+s$ fixed constitute a subgroup G of $\mathfrak{S}_{r+s}$, isomorphic to $\mathfrak{S}_r$. Let S be a subset of $\mathfrak{S}_{r+s}$ containing exactly one element from each left coset $\sigma_0 G$ of $\mathfrak{S}_{r+s}$. Then, since $\epsilon(\sigma_1\sigma_2) = \epsilon(\sigma_1)\, \epsilon(\sigma_2)$,

$$\sum_{\sigma \in \mathfrak{S}_{r+s}} \epsilon(\sigma)\, \sigma \cdot (n_r \otimes d_s) = \sum_{\sigma_0 \in S} \epsilon(\sigma_0) \sum_{\tau \in G} \epsilon(\tau)\, (\sigma_0\tau) \cdot (n_r \otimes d_s).$$

Let $X_i \in \mathfrak{D}^1$ $(1 \leqslant i \leqslant r+s)$, $(Y_1, \ldots, Y_{r+s}) = \sigma_0^{-1}(X_1, \ldots, X_{r+s})$. Then

$$\sum_{\tau \in G} \epsilon(\tau)\, ((\sigma_0\tau) \cdot (n_r \otimes d_s))(X_1, \ldots, X_{r+s})$$
$$= d_s(Y_{r+1}, \ldots, Y_{r+s}) \sum_{\tau \in \mathfrak{S}_r} \epsilon(\tau)\, (\tau \cdot n_r)(Y_1, \ldots, Y_r) = 0.$$

This shows that $b_{r+s} = 0$. Similarly one proves $A_{s+r}(d_s \otimes n_r) = 0$.

For any two $\theta, \omega \in \mathfrak{A}$ we can now define the *exterior product*

$$\theta \wedge \omega = A(\theta \otimes \omega).$$

This turns the $\mathfrak{F}$-module $\mathfrak{A}$ into an associative algebra, isomorphic to $\mathfrak{D}_*/N$. The module $\mathfrak{A}(M)$ of alternate $\mathfrak{F}$-multilinear functions with the exterior multiplication is called the *Grassmann algebra* of the manifold M.

We can also for each $p \in M$ define the Grassmann algebra $\mathfrak{A}(p)$ of the tangent space M_p. The elements of $\mathfrak{A}(p)$ are the alternate, $\boldsymbol{R}$-multilinear, real-valued functions on M_p and the product (also denoted $\wedge$) satisfies

$$\theta_p \wedge \omega_p = (\theta \wedge \omega)_p, \qquad \theta, \omega \in \mathfrak{A}.$$

This turns $\mathfrak{A}(p)$ into an associative algebra containing the dual space M_p^*. If $\theta, \omega \in M_p^*$, we have $\theta \wedge \omega = -\omega \wedge \theta$; as a consequence one derives easily the following rule:

Let $\theta^1, ..., \theta^l \in M_p^*$ and let $\omega^i = \Sigma_{j=1}^l a_{ij}\theta^j$, $1 \leqslant i, j \leqslant l$, $(a_{ij} \in \boldsymbol{R})$. Then

$$\omega^1 \wedge ... \wedge \omega^l = \det(a_{ij})\, \theta^1 \wedge ... \wedge \theta^l.$$

For convenience we write down the exterior multiplication explicitly. Let $f, g \in C^\infty(M)$, $\theta \in \mathfrak{A}_r$, $\omega \in \mathfrak{A}_s$, $X_i \in \mathfrak{D}^1$. Then

$$f \wedge g = fg,$$

$$(f \wedge \theta)(X_1, ..., X_r) = f\,\theta(X_1, ..., X_r),$$

$$(\omega \wedge g)(X_1, ..., X_s) = g\,\omega(X_1, ..., X_s),$$

$$(\theta \wedge \omega)(X_1, ..., X_{r+s}) \tag{4}$$

$$= \frac{1}{(r+s)!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \epsilon(\sigma)\, \theta(X_{\sigma(1)}, ..., X_{\sigma(r)})\, \omega(X_{\sigma(r+1)}, ..., X_{\sigma(r+s)}).$$

We also have the relation

$$\theta \wedge \omega = (-1)^{rs}\omega \wedge \theta. \tag{5}$$

Exterior Differentiation

Let M be a C^∞ manifold, $\mathfrak{A}(M)$ the Grassmann algebra over M. The operator d, the *exterior differentiation*, is described in the following theorem.

Theorem 2.5. *There exists a unique R-linear mapping* $d: \mathfrak{A}(M) \to \mathfrak{A}(M)$ *with the following properties*:

(i) $d\mathfrak{A}_s \subset \mathfrak{A}_{s+1}$ *for each* $s \geqslant 0$.

(ii) *If* $f \in \mathfrak{A}_0$ $(= C^\infty(M))$, *then* df *is the* 1*-form given by* $df(X) = Xf$, $X \in \mathfrak{D}^1(M)$.

(iii) $d \circ d = 0$.

(iv) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2$ *if* $\omega_1 \in \mathfrak{A}_r$, $\omega_2 \in \mathfrak{A}(M)$.

Proof. Assuming the existence of d for M as well as for open submanifolds of M, we first prove a formula for d ((9) below) which then has the uniqueness as a corollary. Let $p \in M$ and $\{x_1, \ldots, x_m\}$ a coordinate system valid on an open neighborhood U of p. Let V be an open subset of U such that $\bar{V}$ is compact and $p \in V$, $\bar{V} \subset U$. From (ii) we see that the forms dx_i $(1 \leqslant i \leqslant m)$ on U satisfy $dx_i(\partial/\partial x_j) = \delta_{ij}$ on U. Hence dx_i $(1 \leqslant i \leqslant m)$ is a basis of the $C^\infty(U)$-module $\mathfrak{D}_1(U)$; thus each element in $\mathfrak{D}_*(U)$ can be expressed in the form

$$\sum F_{i_1 \ldots i_r}\, dx_{i_1} \otimes \ldots \otimes dx_{i_r}, \qquad F_{i_1 \ldots i_r} \in C^\infty(U).$$

It follows that if $\theta \in \mathfrak{A}(M)$ and if θ_U denotes the form induced by θ on U, then θ_U can be written

$$\theta_U = \sum f_{i_1 \ldots i_r}\, dx_{i_1} \wedge \ldots \wedge dx_{i_r}, \qquad f_{i_1 \ldots i_r} \in C^\infty(U). \tag{6}$$

This is called an *expression* of θ_U on U. We shall prove the formula

$$d(\theta_V) = (d\theta)_V.$$

Owing to Lemma 1.2 there exist functions $\psi_{i_1 \ldots i_r} \in C^\infty(M)$, $\varphi_i \in C^\infty(M)$ $(1 \leqslant i \leqslant m)$ such that

$$\psi_{i_1 \ldots i_r} = f_{i_1 \ldots i_r}, \qquad \varphi_1 = x_1, \ldots, \varphi_m = x_m \text{ on } V.$$

We consider the form

$$\omega = \sum \psi_{i_1 \ldots i_r}\, d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_r}$$

on M. We have obviously $\omega_V = \theta_V$. Moreover, since $d(f(\theta - \omega)) = df \wedge (\theta - \omega) + fd(\theta - \omega)$ for each $f \in C^\infty(M)$, we can, choosing f identically 0 outside V, identically 1 on an open subset of V, deduce that $(d\theta)_V = (d\omega)_V$.

Since

$$d\omega = \sum d\psi_{i_1 \ldots i_r} \wedge d\varphi_{i_1} \wedge \ldots \wedge d\varphi_{i_r}$$

owing to (iii) and (iv), and since $d(f_V) = (df)_V$ for each $f \in C^\infty(M)$, we conclude that

$$(d\omega)_V = \sum df_{i_1 \ldots i_r} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r}. \tag{7}$$

This proves the relation

$$(d\theta)_V = d(\theta_V) = \sum df_{i_1 \ldots i_r} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_r}. \tag{8}$$

On M itself we have the formula

$$\begin{aligned}(p+1)\, d\omega(X_1, \ldots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} X_i \cdot \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}) \quad (9)\end{aligned}$$

for $\omega \in \mathfrak{A}_p(M)$ $(p \geqslant 1)$, $X_i \in \mathfrak{D}^1(M)$. In fact, it suffices to prove it in a coordinate neighborhood of each point; in that case it is a simple consequence of (8). The uniqueness of d is now obvious.

On the other hand, to prove the existence of d, we *define* d by (9) and (ii). Using the relation $[X, fY] = f[X, Y] + (Xf)\, Y$ $(f \in \mathfrak{F};\ X, Y \in \mathfrak{D}^1)$, it follows quickly that the right-hand side of (9) is $\mathfrak{F}$-linear in each variable X_i and vanishes whenever two variables coincide. Hence $d\omega \in \mathfrak{A}_{p+1}$ if $\omega \in \mathfrak{A}_p$. If $X \in \mathfrak{D}^1$, let X_V denote the vector field induced on V. Then $[X, Y]_V = [X_V, Y_V]$ and therefore the relation $(d\theta)_V = d(\theta_V)$ follows from (9). Next we observe that (8) follows from (9) and (ii). Also

$$d(fg) = f dg + g df \tag{10}$$

as a consequence of (ii). To show that (iii) and (iv) hold, it suffices to show that they hold in a coordinate neighborhood of each point of M. But on V, (iv) is a simple consequence of (10) and (8). Moreover, (8) and (ii) imply $d(dx_i) = 0$; consequently (using (iv)),

$$d(df) = d\left(\sum_j \frac{\partial f}{\partial x_j}\, dx_j\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\, dx_i \wedge dx_j = 0$$

for each $f \in C^\infty(U)$. The relation (iii) now follows from (8) and (iv).

Let M and N be C^∞ manifolds and $\Phi : M \to N$ a differentiable mapping. Let ω be an r-form on N. Then we can define an r-form $\Phi^*\omega$ on M which satisfies

$$\Phi^*\omega(X_1, ..., X_r) = \omega(Y_1, ..., Y_r) \circ \Phi$$

whenever the vector fields X_i and Y_i $(1 \leqslant i \leqslant r)$ are Φ-related. It suffices to put

$$(\Phi^*\omega)_p(A_1, ..., A_r) = \omega_{\Phi(p)}(d\Phi_p(A_1), ..., d\Phi_p(A_r))$$

for each $p \in M$, and $A_i \in M_p$. If $f \in C^\infty(N)$, we put $\Phi^*f = f \circ \Phi$ and by linearity $\Phi^*\theta$ is defined for each $\theta \in \mathfrak{A}(M)$. Then the following formulas hold:

$$\Phi^*(\omega_1 \wedge \omega_2) = \Phi^*(\omega_1) \wedge \Phi^*(\omega_2), \qquad \omega_1, \omega_2 \in \mathfrak{A}(M); \tag{5}$$

$$d(\Phi^*\omega) = \Phi^*(d\omega). \tag{6}$$

In fact, (5) follows from (4), § 2, and (6) is proved below. In the same way we can define Φ^*T for an arbitrary covariant tensor field $T \in \mathfrak{D}_*(M)$. If $M = N$ and Φ is a diffeomorphism of M onto itself such that $\Phi^*T = T$, we say that T is ***invariant*** under Φ.

The computation of $\Phi^*\omega$ in coordinates is very simple. Suppose U and V are open sets in M and N, respectively, where the coordinate systems

$$\xi: q \to (x_1(q), ..., x_m(q)), \qquad \eta: r \to (y_1(r), ..., y_n(r))$$

are valid. Assume $\Phi(U) \subset V$. On U, Φ has a coordinate expression

$$y_j = \varphi_j(x_1, ..., x_m) \qquad (1 \leqslant j \leqslant n).$$

If $\omega \in \mathfrak{A}(N)$, the form ω_V has an expression

$$\omega_V = \sum g_{j_1 ... j_s} \, dy_{j_1} \wedge ... \wedge dy_{j_s} \tag{7}$$

where $g_{j_1 ... j_s} \in C^\infty(V)$. The form $\Phi^*\omega$ induces the form $(\Phi^*\omega)_U$ on U, which has an expression

$$(\Phi^*\omega)_U = \sum f_{i_1 ... i_r} \, dx_{i_1} \wedge ... \wedge dx_{i_r}.$$

This expression is obtained just by substituting

$$y_j = \varphi_j(x_1, ..., x_m), \qquad dy_j = \sum_{i=1}^{m} \frac{\partial \varphi_j}{\partial x_i} \, dx_i \qquad (1 \leqslant j \leqslant n)$$

into (7). This follows from (5) if we observe that (2) implies

$$\Phi^*(dy_j) = \sum_{i=1}^{m} \left(\frac{\partial \varphi_j}{\partial x_i} \circ \xi \right) dx_i.$$

This proves (6) if ω is a function, hence, by (7), in general.

Let V be a finite-dimensional vector space and $Z_1, \ldots, Z_n$ a basis of V. In order that a bilinear map $(X, Y) \to [X, Y]$ of $V \times V$ into V turn V into a Lie algebra it is necessary and sufficient that the ***structural constants*** $\gamma^i{}_{jk}$ given by

$$[Z_j, Z_k] = \sum_1^n \gamma^i{}_{jk} Z_i$$

satisfy the conditions

$$\gamma^i{}_{jk} + \gamma^i{}_{kj} = 0$$

$$\sum_{j=1}^n \gamma^i{}_{jl}\gamma^j{}_{km} + \gamma^i{}_{jm}\gamma^j{}_{lk} + \gamma^i{}_{jk}\gamma^j{}_{ml} = 0.$$

Theorem 2.6. (the third theorem of Lie) *Let $c^i{}_{jk} \in \boldsymbol{R}$ be constants $(1 \leqslant i, j, k \leqslant n)$ satisfying the relations*

$$c^i{}_{jk} + c^i{}_{kj} = 0 \tag{8}$$

$$\sum_{j=1}^n (c^i{}_{jl}c^j{}_{km} + c^i{}_{jm}c^j{}_{lk} + c^i{}_{jk}c^j{}_{ml}) = 0. \tag{9}$$

Then there exist an open neighborhood N of 0 in $\boldsymbol{R}^n$ and a basis $Y_1, \ldots, Y_n$ of $\mathfrak{D}^1(N)$ over $C^\infty(N)$ satisfying the relations

$$[Y_j, Y_k] = \sum_{i=1}^n c^i{}_{jk} Y_i. \tag{10}$$

Proof. We shall find a basis $\omega_1, \ldots, \omega_n$ of $\mathfrak{D}_1(N)$ over $C^\infty(N)$ satisfying the relations

$$d\omega_i = -\tfrac{1}{2} \sum_{j,k=1}^n c^i{}_{jk}\omega_j \wedge \omega_k. \tag{11}$$

Then the $Y_1, \ldots, Y_n$ can be chosen as the basis dual to $\omega_1, \ldots, \omega_n$ (Lemma 2.3, Chapter I), and (10) follows from (11).

We start by *defining* 1-forms

$$\theta_i = \theta_i(t, a_1, \ldots, a_n) = \sum_{j=1}^n f_{ij}(t, a_1, \ldots, a_n)\, da_j$$

as solutions to the differential equations

$$\frac{\partial\theta_i}{\partial t} = da_i - \sum_{j,k} c^i{}_{jk} a_j \theta_k, \qquad \theta_i(0, a_1, \ldots, a_n) = 0. \tag{13}$$

This amounts to a linear inhomogeneous constant coefficient system of differential equations for the functions f_{ij}, so these functions are uniquely determined for $(t, a_1, \ldots, a_n) \in \boldsymbol{R}^{n+1}$. Using (13) we get

$$\begin{aligned} d\theta_i &= \sum_j \frac{\partial f_{ij}}{\partial t}\, dt \wedge da_j + \sum_{j,k} \frac{\partial f_{ij}}{\partial a_k}\, da_k \wedge da_j \\ &= \Big(-da_i + \sum_{j,k} c^i{}_{jk} a_j \theta_k\Big) \wedge dt + \sum_{j,k} \frac{\partial f_{ij}}{\partial a_k}\, da_k \wedge da_j. \end{aligned}$$

We write this formula

$$d\theta_i = \alpha_i \wedge dt + \beta_i, \tag{14}$$

where the α_i and β_i are 1-forms and 2-forms, respectively, which do not contain dt. Next we put

$$\sigma_i = \beta_i + \tfrac{1}{2} \sum_{j,k} c^i{}_{jk} \theta_j \wedge \theta_k$$

and since we would by Chapter I expect the forms $\theta_i(t, a_1, ..., a_n)_{t=1}$ to satisfy (11) we now try to prove that $\sigma_i = 0$. Using (8), and writing $\cdots$ for terms which do not contain dt, we have

$$\begin{aligned} d\sigma_i &= d\beta_i + \sum_{j,k} c^i{}_{jk} d\theta_j \wedge \theta_k \\ &= -dt \wedge d\alpha_i - dt \wedge \sum_{j,k} c^i{}_{jk} \alpha_j \wedge \theta_k + ... \end{aligned}$$

Using the expression for α_i and (14), this becomes

$$\begin{aligned} &-dt \wedge \sum_{j,k} c^i{}_{jk}(da_j \wedge \theta_k + a_j\beta_k + \alpha_j \wedge \theta_k) + ... \\ &\qquad = -dt \wedge \sum_{j,k} c^i{}_{jk} \Big(\sum_{pq} c^j{}_{pq} a_p \theta_q \wedge \theta_k + a_j \beta_k \Big) + ... \end{aligned}$$

But since $\theta_q \wedge \theta_k = -\theta_k \wedge \theta_q$, we have

$$\sum_{j,k,q} c^i{}_{jk} c^j{}_{pq} \theta_q \wedge \theta_k = \tfrac{1}{2} \sum_{j,k,q} (c^i{}_{jk} c^j{}_{pq} - c^i{}_{jq} c^j{}_{pk}) \theta_q \wedge \theta_k,$$

which by (8) and (9) equals

$$-\tfrac{1}{2} \sum_{j,k,q} c^i{}_{jp} c^j{}_{qk} \theta_q \wedge \theta_k.$$

This proves

$$\begin{aligned} d\sigma_i &= -dt \wedge \Big(\sum_{j,k} c^i{}_{jk} a_j \beta_k - \tfrac{1}{2} \sum_{j,k,p,q} a_p c^i{}_{jp} c^j{}_{qk} \theta_q \wedge \theta_k \Big) + ... \\ &= -dt \wedge \sum_{j,k} c^i{}_{jk} \Big(-a_k \beta_j - \tfrac{1}{2} \sum_{qr} a_k c^j{}_{qr} \theta_q \wedge \theta_r \Big) + ... \end{aligned}$$

so

$$d\sigma_i = dt \wedge \sum_{jk} c^i{}_{jk} a_k \sigma_j + \dots . \tag{15}$$

This amounts to

$$\frac{\partial \sigma_i}{\partial t} = \sum_{jk} c^i{}_{jk} a_k \sigma_j$$

which, since the σ_j all vanish for $t = 0$, implies that each σ_i vanishes identically. Thus we see from (14) that the forms $\omega_i = \theta_i\,(1, a_1, ..., a_n)$ will satisfy (11). Finally, (13) implies that

$$\theta_i(t, 0, ..., 0) = t\, da_i,$$

so the forms ω_i are linearly independent at $(a_1, ..., a_n) = (0, ..., 0)$, hence also in a suitable neighborhood of the origin. This concludes the proof.

10 Integration of Forms

A manifold M is said to be *orientable* if there exists a collection $(U_\alpha, \psi_\alpha)_{\alpha\in A}$ of local charts such that $\{U_\alpha\}_{\alpha\in A}$ is a covering of M and such that for any $\alpha, \beta \in A$, the mapping $\psi_\beta \circ \psi_\alpha^{-1}$ has strictly positive Jacobian on its domain of definition $\psi_\alpha(U_\alpha \cap U_\beta)$. The manifold M is said to be *oriented* if such a collection $(U_\alpha, \psi_\alpha)_{\alpha\in A}$ has been chosen.

For the definition of integration on M the following weak form of partition of unity is useful.

Lemma 10.1. *Let $K \subset M$ be compact and $V_1, \dots, V_n$ open sets such that $K \subset \bigcup_1^n V_j$. Then there exist functions $\varphi_j \in \mathcal{C}_c^\infty(V_j)$ such that $\varphi_j \geq 0$ and $\sum_1^n \varphi_j \leq 1$ with equality sign holding in a neighborhood of K.*

Proof:

For each j there exists a compact set $C_j \subset V_j$. Using Lemma 1.2 in Ch. I we can pick a function $\psi_j \in \mathcal{C}_c^\infty(V_j)$ such that $0 \leq \psi_j \leq 1$ and $\psi_j = 1$ in a neighborhood of C_j. **We can also arrange that $K \subset \bigcup_j C_j$. Then**

$$\varphi_1 = \psi_1, \dots, \varphi_j = \psi_j(1-\psi_1)\dots(1-\psi_{j-1})$$

has the desired property.

Let S be a locally compact Hausdorff space. The set of real-valued continuous functions on S will be denoted by $C(S)$, and $C_c(S)$ shall denote the set of functions in $C(S)$ of compact support. A *measure* on S is by definition a linear mapping $\mu: C_c(S) \to \boldsymbol{R}$ with the property that for each compact subset $K \subset S$ there exists a constant M_K such that

$$|\mu(f)| \leq M_K \sup_{x\in S} |f(x)|$$

for all $f \in C_c(S)$ whose support is contained in K. We recall that a linear mapping $\mu: C_c(S) \to \boldsymbol{R}$ which satisfies $\mu(f) \geq 0$ for $f \geq 0$, $f \in C_c(S)$, is a measure on S. Such a measure is called a *positive measure.* For a manifold M we put $\mathscr{E}(M) = C^\infty(M)$, $\mathscr{D}(M) = \mathscr{E}(M) \cap C_c(M)$. Suppose M is an orientable m-dimensional manifold and let $(U_\alpha, \phi_\alpha)_{\alpha\in A}$ be a collection of local charts on M by which M is oriented.

Let ω be an m-form on M. We shall define the integral $\int_M f\omega$ for each $f \in C_c(M)$. First we assume that f has compact support contained in a coordinate neighborhood U_α and let

$$\phi_\alpha(q) = (x_1(q), \ldots, x_m(q)), \qquad q \in U_\alpha.$$

On U_α, ω has an "expression" (cf., [DS], Chapter I, §2, No. 4)

$$\omega_{U_\alpha} = F_\alpha(x_1, \ldots, x_m)\, dx_1 \wedge \cdots \wedge dx_m, \tag{1}$$

and we set

$$\int_M f\omega = \int_{\phi_\alpha(U_\alpha)} (f \circ \phi_\alpha^{-1})(x_1, \ldots, x_m) F_\alpha(x_1, \ldots, x_m)\, dx_1 \cdots dx_m.$$

On using the transformation formula for multiple integrals we see that if f has compact support inside the intersection $U_\alpha \cap U_\beta$ of two coordinate neighborhoods, then the right-hand side in the formula above is

$$\int_{\phi_\beta(U_\beta)} (f \circ \phi_\beta^{-1})(y_1, \ldots, y_m) F_\beta(y_1, \ldots, y_m)\, dy_1 \cdots dy_m,$$

if $F_\beta\, dy_1 \wedge \cdots \wedge dy_m$ is the expression for ω on U_β. Thus $\int_M f\omega$ is well-defined. Next, let f be an arbitrary function in $C_c(M)$. Then f vanishes

outside a compact subset of M so by Lemma 10.1, f can be expressed as a finite sum $f = \sum_i fi$

each f_i has compact support inside some neighborhood U_α from our covering. We put

$$\int_M f\omega = \sum_i \int_M f_i\omega.$$

Here it has to be verified that the right-hand side is independent of the chosen decomposition $f = \sum_i f_i$ of f. Let $f = \sum_j g_j$ be another such decomposition and select $\phi \in C_c(M)$ such that $\phi = 1$ on the union of the supports of all f_i and g_j. Then $\phi = \sum \phi_\alpha$ (finite sum), where each ϕ_α has support inside a coordinate neighborhood from our covering. We have

$$\sum_i f_i \phi_\alpha = \sum_j g_j \phi_\alpha,$$

and since each summand has support inside a fixed coordinate neighborhood,

$$\sum_i \int (f_i \phi_\alpha)\omega = \sum_j \int (g_j \phi_\alpha)\omega.$$

For the same reason the formulas

$$f_i = \sum_\alpha f_i \phi_\alpha, \qquad g_j = \sum_\alpha g_j \phi_\alpha$$

imply that

$$\int f_i\omega = \sum_\alpha \int (f_i\phi_\alpha)\omega, \qquad \int g_j\omega = \sum_\alpha \int (g_j\phi_\alpha)\omega,$$

from which we derive the desired relation

$$\sum_i \int f_i\omega = \sum_j \int g_j\omega.$$

The integral $\int f\omega$ is now well-defined and the mapping

$$f \to \int_M f\omega, \qquad f \in C_c(M),$$

is a measure on M. We have obviously

Lemma 10.2. *If $\int_M f\omega = 0$ for all $f \in C_c(M)$, then $\omega = 0$.*

Definition. The m-form ω is said to be *positive* if for each $\alpha \in A$, the function F_α in (1) is >0 on $\phi_\alpha(U_\alpha)$.

If ω is a positive m-form on M, then it follows readily from ~~Theorem~~ **Lemma 10.1 proved above** that $\int_M f\omega \geq 0$ for each nonnegative function $f \in C_c(M)$. Thus, a positive m-form gives rise to a positive measure.

Suppose M and N are two oriented manifolds and let Φ be a diffeomorphism of M onto N. We assume that Φ is *orientation preserving*, that is, if the collection of local charts $(U_\alpha, \phi_\alpha)_{\alpha\in A}$ defines the orientation on M, then the collection $(\Phi(U_\alpha), \phi_\alpha \circ \Phi^{-1})_{\alpha\in A}$ of local charts on N defines the orientation on N. Let m denote the dimension of M and N.

Let ω be an m-form on N and $\Phi^*\omega$ its *transform* or *pullback* by Φ **(last section)**. Then the formula

$$\int_M f\Phi^*\omega = \int_N (f \circ \Phi^{-1})\omega \tag{2}$$

holds for all $f \in C_c(M)$. In fact, it suffices to verify (2) in the case when f has compact support inside a coordinate neighborhood U_α. If we evaluate the left-hand side of (2) by means of the coordinate system ϕ_α and the right-hand side of (2) by means of the coordinate system $\phi_\alpha \circ \Phi^{-1}$, both sides of (2) reduce to the same integral.

§7. Invariant Differential Forms

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A differential form ω on G is called *left invariant* if $L(x)^*\,\omega = \omega$ for all $x \in G$, $L(x)$ denoting the left translation $g \to xg$ on G. Similarly we define *right invariant* differential forms on G. A form is called *bi-invariant* if it is both left and right invariant.

Let ω be a left invariant p-form on G. Then if $X_1, \ldots, X_{p+1} \in \mathfrak{g}$ are arbitrary, $\tilde{X}_i$ the corresponding left invariant vector fields on G, we have by (9), Chapter I, §2,

$$(p+1)\, d\omega(\tilde{X}_1, \ldots, \tilde{X}_{p+1})$$

$$= \sum_{i<j} (-1)^{i+j}\, \omega([\tilde{X}_i, \tilde{X}_j], \tilde{X}_1, \ldots, \hat{\tilde{X}}_i, \ldots, \hat{\tilde{X}}_j, \ldots, \tilde{X}_{p+1}). \tag{1}$$

Lemma 7.1. *Let ω be a left invariant form on G. If ω is right invariant then ω is closed, that is, $d\omega = 0$.*

Proof: Let ω be a p-form, J the mapping $x \to x^{-1}$. Then the pull-back $J^*\omega$ is still bi-invariant and $J^*\omega = (-1)^p\omega$. Now $d\omega$ is also bi-invariant and $= dJ^*\omega = J^*\, d\omega = (-1)^{p+1}\, d\omega$. Since the left hand side equals $(-1)^p\, d\omega$ we have $d\omega = 0$.

Proposition 7.2. *Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ and $\omega_1, \ldots, \omega_n$ the 1-forms on G determined by $\omega_i(\tilde{X}_j) = \delta_{ij}$. Then*

$$d\omega_i = -\tfrac{1}{2} \sum_{j,k=1}^{n} c^i{}_{jk}\, \omega_j \wedge \omega_k \tag{3}$$

if $c^i{}_{jk}$ are the structural constants given by

$$[X_j, X_k] = \sum_{i=1}^{n} c^i{}_{jk} X_i.$$

Equations (3) are known as the Maurer-Cartan equations. They follow immediately from (1). Note that the Jacobi identity for $\mathfrak{g}$ is reflected in the relation $d^2 = 0$.

Example. Consider as in §1 the general linear group $GL(n, \boldsymbol{R})$ with the usual coordinates $\sigma \to (x_{ij}(\sigma))$. Writing $X = (x_{ij})$, $dX = (dx_{ij})$, the matrix

$$\Omega = X^{-1}\, dX,$$

whose entries are 1-forms on G, is invariant under left translations $X \to \sigma X$ on G. Writing

$$dX = X\Omega,$$

we can derive

$$0 = (dX) \wedge \Omega + X \wedge d\Omega,$$

where $\wedge$ denote the obvious wedge product of matrices. Multiplying by X^{-1}, we obtain

$$d\Omega + \Omega \wedge \Omega = 0, \tag{4}$$

which is an equivalent form of (3).

More generally, consider for each x in the Lie group G the mapping

$$dL(x^{-1})_x : G_x \to \mathfrak{g}$$

and let Ω denote the family of these maps. In other words,

$$\Omega_x(v) = dL(x^{-1})(v) \qquad \text{if} \quad v \in G_x.$$

Then Ω is a 1-form on G with values in $\mathfrak{g}$. Moreover, if $x, y \in G$, then

$$\Omega_{xy} \circ dL(x)_y = \Omega_y,$$

so Ω is left invariant. Thus $\Omega_x = \sum_{i=1}^{n} (\theta_i)_x X_i$ in terms of the basis $X_1, ..., X_n$ in Prop. 7.2, $\theta_1, ..., \theta_n$ being left invariant 1-forms on G. But applying Ω_x to the vectors $(\tilde{X}_j)_x$ it is clear that $\theta_j = \omega_j$ $(1 \leqslant j \leqslant n)$. Hence we write

$$\Omega = \sum_{i=1}^{n} \omega_i X_i, \qquad d\Omega = \sum_{i=1}^{n} d\omega_i X_i.$$

If θ is any $\mathfrak{g}$-valued 1-form on a manifold X, we can define $[\theta, \theta]$ as the 2-form with values in $\mathfrak{g}$ given by

$$[\theta, \theta]_x (v_1, v_2) = [\theta_x(v_1), \theta_x(v_2)], \qquad x \in X, \quad v_1, v_2 \in X_x.$$

Then Prop. 7.2 can be reformulated as follows.

Proposition 7.3. *Let Ω denote the unique left invariant $\mathfrak{g}$-valued 1-form on G such that Ω_e is the identity mapping of G_e into $\mathfrak{g}$. Then*

$$d\Omega + \tfrac{1}{2}[\Omega, \Omega] = 0.$$

In fact, since $c^i{}_{jk}$ is skew in (j, k)

$$\begin{aligned}
[\Omega, \Omega]_x (v_1, v_2) &= \Big[\sum_j \omega_j(v_1) X_j, \sum_k \omega_k(v_2) X_k\Big] \\
&= \sum_{i,j,k} \omega_j(v_1)\, \omega_k(v_2)\, c^i{}_{jk} X_i = \sum_{i,j,k} c^i{}_{jk} (\omega_j \wedge \omega_k)(v_1, v_2)\, X_i \\
&= -2(d\Omega)_x (v_1, v_2).
\end{aligned}$$

We shall now determine the Maurer-Cartan forms ω_i explicitly in terms of the structural constants $c^i{}_{jk}$. Since exp is a C^∞ map from $\mathfrak{g}$ into G, the forms $\exp^* \omega_i$ can be expressed in terms of the Cartesian coordinates $(x_1, ..., x_n)$ of $\mathfrak{g}$ with respect to the basis $X_1, ..., X_n$,

$$(\exp^*(\omega_i))_X (X_j) = A_{ij}(x_1, ..., x_n), \tag{5}$$

where $X = \Sigma_i x_i X_i$ and $A_{ij} \in C^\infty(R^n)$. Now let N_0 be an open star-shaped neighborhood of 0 in $\mathfrak{g}$ which exp maps diffeomorphically onto an open neighborhood N_e of e in G. Then $(x_1, \ldots, x_n)$ are canonical coordinates of $x = \exp X$ $(X \in N_0)$ with respect to the basis $X_1, \ldots, X_n$. Then, if $f \in C^\infty(G)$,

$$d \exp_X(X_j) f = (X_j)_X (f \circ \exp) = \left(\frac{d}{dt} f(\exp(X + tX_j))\right)_{t=0},$$

whence

$$d \exp_X(X_j) = \frac{\partial}{\partial x_j}.$$

Consequently,

$$(\omega_i)_x \left(\frac{\partial}{\partial x_j}\right) = \omega_i(d \exp_X(X_j)) = \exp^*(\omega_i)_X (X_j),$$

so

$$(\omega_i)_x = \sum_{j=1}^{n} A_{ij}(x_1, \ldots, x_n)\, dx_j. \tag{6}$$

Thus by Theorem 1.7 and the left invariance of ω_i,

$$A_{ij}(x_1, \ldots, x_n) = (\omega_i)_x (d \exp_X(X_j)) = (\omega_i)_e \left(\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X} (X_j)\right).$$

Summarizing, we have proved the following result.

Theorem 7.4. *Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ and the left-invariant 1-forms ω_i determined by $\omega_i(\tilde{X}_j) = \delta_{ij}$. Then the functions A_{ij} in* (5) *and* (6) *are given by the structural constants as follows. For $X = \Sigma_i x_i X_i$ in $\mathfrak{g}$ let $A(X)$ be defined by*

$$A(X)(X_j) = \sum_i A_{ij}(x_1, \ldots, x_n)\, X_i \qquad (1 \leqslant j \leqslant n).$$

Then

$$A(X) = \frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X} \tag{7}$$

and

$$\operatorname{ad} X(X_j) = \sum_k \left(\sum_i x_i c^k{}_{ij}\right) X_k.$$

§ 1. Invariant Measures on Coset Spaces*

Let M be a manifold and Φ a diffeomorphism of M onto itself. We recall that a differential form ω on M is called invariant under Φ if

$$\Phi^*\omega = \omega.$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A differential form ω on G is called *left-invariant* if $L_x^*\omega = \omega$ for all $x \in G$, L_x [or $L(x)$] denoting the left translation $g \to xg$ on G. Also, R_x [or $R(x)$] denotes the right translation $g \to gx$ on G and *right-invariant* differential forms on G can be defined. If $X \in \mathfrak{g}$, let $\tilde{X}$ denote the corresponding left-invariant vector field on G. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$. The equations $\omega^i(\tilde{X}_j) = \delta^i_j$ determine uniquely n 1-forms ω^i on G. These are clearly left-invariant and the exterior product $\omega = \omega^1 \wedge \cdots \wedge \omega^n$ is a left-invariant n-form on G. Each 1- form on G can be written $\sum_{i=1}^n f_i\omega^i$, where $f_i \in \mathscr{E}(G)$; it follows that each n-form can be written $f\omega$, where $f \in \mathscr{E}(G)$. Thus, except for a constant factor, ω is the only left-invariant n-form on G. Let

$$\phi: x \to (x_1(x), \ldots, x_n(x))$$

* Note that 1.1, 1.2, 1.3 , 1.11, 1.12, and 1.13 do not appear in this section.

be a system of canonical coordinates with respect to the basis $X_1, \ldots, X_n$ of $\mathfrak{g}$, valid on a connected open neighborhood U of e (cf. [DS], Chapter II, §1). On U, ω has an expression

$$\omega_U = F(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$$

and $F > 0$. Now, if $g \in G$, the pair $(L_g U, \phi \circ L_{g^{-1}})$ is a local chart on a connected neighborhood of g. We put $(\phi \circ L_{g^{-1}})(x) = (y_1(x), \ldots, y_n(x))$ $(x \in L_g U)$. Since $y_i(gx) = x_i(x)$ $(x \in U \cap L_g U)$, the mapping

$$L_g : U \to L_g U$$

has coordinate expression given by

$$(y_1, \ldots, y_n) = (x_1, \ldots, x_n).$$

On $L_g U$, ω has an expression

$$\omega_{L_g U} = G(y_1, \ldots, y_n)\, dy_1 \wedge \cdots \wedge dy_n,$$

so that the invariance condition $\omega_x = L_g^* \omega_{gx}$ $(x \in U \cap L_g U)$ can be written

$$\begin{aligned} G(y_1(x), \ldots, y_n(x))&(dy_1 \wedge \cdots \wedge dy_n)_x \\ &= G(x_1(x), \ldots, x_n(x))(dx_1 \wedge \cdots \wedge dx_n)_x. \end{aligned}$$

Hence $F(x_1(x), \ldots, x_n(x)) = G(x_1(x), \ldots, x_n(x))$ and

$$F(x_1(x), \ldots, x_n(x)) = F(y_1(x), \ldots, y_n(x)) \frac{\partial(y_1(x), \ldots, y_n(x))}{\partial(x_1(x), \ldots, x_n(x))}$$

for $x \in U \cap L_g U$, which shows that the Jacobian of $(\phi \circ L_{g^{-1}}) \circ \phi^{-1}$ is >0. Consequently, the collection $(L_g U, \phi \circ L_{g^{-1}})_{g \in G}$ of local charts turns G into an oriented manifold and each left translation is orientation preserving. The orientation of G depends on the choice of basis of $\mathfrak{g}$. If $X_1', \ldots, X_n'$ is another basis, then the resulting orientation of G is the same as that before if and only if the linear transformation

$$X_i \to X_i' \qquad (1 \leq i \leq n)$$

has positive determinant.

The form ω is a positive left-invariant n-form on G and except for a constant positive factor, ω is uniquely determined by these properties. We shall denote it by $d_l g$. The linear mapping of $C_c(G)$ into $\boldsymbol{R}$ given by $f \to \int f\, d_l g$ is a *measure* on G, which we denote by μ_l. This measure is positive; moreover, it is *left-invariant* in the sense that $\mu_l(f \circ L_x) = \mu_l(f)$ for $x \in G$, $f \in C_c(G)$.

Similarly, G can be turned into an oriented manifold such that each R_g $(g \in G)$ is orientation preserving. There exists a right-invariant positive n-form $d_r g$ on G and this is unique except for a constant positive factor. We define the *right-invariant* positive measure μ_r on G by

$$\mu_r(f) = \int f\, d_r g, \qquad f \in C_c(G).$$

The group G has been oriented in two ways. The left-invariant orientation is invariant under all right translations R_x $(x \in G)$ if and only if it is invariant under all $I(x) = L_x \circ R_{x^{-1}}$ $(x \in G)$. Since the differential $dI(x)_g$ satisfies

$$dI(x)_g = dL_{xgx^{-1}} \circ \mathrm{Ad}(x) \circ dL_{g^{-1}},$$

the necessary and sufficient condition is $\det \mathrm{Ad}(x) > 0$ for all $x \in G$. This condition is always fulfilled if G is connected.

Lemma 1.4. *With the notation above we have*

$$d_r g = c \det \mathrm{Ad}(g)\, d_l g,$$

where c is a constant.

Proof. Let $\theta = \det \mathrm{Ad}(g)\, d_l g$ and let $x \in G$. Then

$$(R_{x^{-1}})^*\theta = \det \mathrm{Ad}(gx^{-1})(R_{x^{-1}})^*\, d_l g = \det \mathrm{Ad}(gx^{-1}) I(x)^*\, d_l g.$$

At the point $g = e$ we have

$$(I(x)^*(d_l g))_e = \det \mathrm{Ad}(x)(d_l g)_e.$$

Consequently,

$$(R_{x^{-1}}{}^*\theta)_e = \det \mathrm{Ad}(e)(d_l g)_e = \theta_e.$$

Thus, θ is right-invariant and therefore proportional to $d_r g$.

Remark. If G is connected it can be oriented in such a way that all left and right translations are orientation preserving. If $d_r g$ and $d_l g$ are defined by means of this orientation, Lemma 1.4 holds with $c > 0$.

Corollary 1.5. *Let $x, y \in G$ and put $d_l(ygx) = (L_y R_x)^*\, d_l g$, $d_r(xgy) = (L_x R_y)^*\, d_r g$. Moreover, if J denotes the mapping $g \to g^{-1}$, put $d_l(g^{-1}) = J^*(d_l g)$. Then*

$$d_l(gx) = \det \mathrm{Ad}(x^{-1})\, d_l(g), \qquad d_r(xg) = \det \mathrm{Ad}(x)\, d_r g,$$
$$d_l(g^{-1}) = (-1)^{\dim G} \det \mathrm{Ad}(g)\, d_l g.$$

In fact, the lemma implies that

$$c \det \mathrm{Ad}(g)\, d_l g = d_r g = d_r(gx) = c \det \mathrm{Ad}(gx)\, d_l(gx),$$
$$d_r(xg) = c \det \mathrm{Ad}(xg)\, d_l(xg) = c \det \mathrm{Ad}(xg)\, d_l g.$$

Finally, since $JR_x = L_{x^{-1}}J$, we have

$$(R_x)^* d_l(g^{-1}) = (R_x)^* J^* d_l g = (JR_x)^* d_l g = (L_{x^{-1}}J)^* d_l g = J^* d_l g,$$

so that $d_l(g^{-1})$ is right-invariant, hence proportional to $d_r g$. But obviously

$$(d_l(g^{-1}))_e = (-1)^{\dim G}(d_l g)_e,$$

so that the corollary is verified.

Definition. A Lie group G is called *unimodular* if the left invariant measure μ_l is also right-invariant.

In view of Corollary 1.5 we have by (2)

$$\mu_l(f \circ R_x) = |\det \mathrm{Ad}(x)|\, \mu_l(f). \tag{6}$$

It follows that G is unimodular if and only if $|\det \mathrm{Ad}(x)| = 1$ for all $x \in G$. If this condition is satisfied, the measures μ_l and μ_r coincide except for a constant factor.

Proposition 1.6. *The following Lie groups are unimodular:*

(i) *Lie groups G for which* $\mathrm{Ad}(G)$ *is compact;*

(ii) *semisimple Lie groups;*

(iii) *connected nilpotent Lie groups.*

Proof. In the case (i), the group $\{|\det \mathrm{Ad}(x)| : x \in G\}$ is a compact subgroup of the multiplicative group of positive real numbers. This subgroup necessarily consists of one element, so that G is unimodular. In the case (ii), each $\mathrm{Ad}(x)$ leaves invariant a nondegenerate bilinear form (namely, the Killing form). It follows that $(\det \mathrm{Ad}(x))^2 = 1$. Finally, let N be a connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. If $X \in \mathfrak{n}$, then $\mathrm{ad}\, X$ is nilpotent, so that $\mathrm{Tr}(\mathrm{ad}\, X) = 0$. Since

$$\det e^A = r^{\mathrm{Tr}\, A}$$

for an arbitrary linear transformation A, we obtain

$$\det \mathrm{Ad}(\exp X) = e^{\mathrm{Tr}(\mathrm{Ad}\, X)} = 1.$$

This proves (iii).

Notation. In the sequel we shall mostly use the left invariant measure μ_l. The measure dg is usually called *Haar measure* on G. For simplicity we shall write μ instead of μ_l and dg instead of $d_l g$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$; let H be a closed subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Each $x \in G$ gives rise to an analytic diffeomorphism $\tau(x): gH \to xgH$ of G/H onto itself. Let π denote the natural mapping of G onto G/H and put $o = \pi(e)$. If $h \in H$, $(d\tau(h))_o$ is an endomorphism of the tangent space $(G/H)_o$. For simplicity, we shall write $d\tau(h)$ instead of $(d\tau(h))_o$ and $d\pi$ instead of $(d\pi)_e$.

Lemma 1.7.

$$\det(d\tau(h)) = \frac{\det \mathrm{Ad}_G(h)}{\det \mathrm{Ad}_H(h)} \qquad (h \in H).$$

Proof. It was shown in [DS], Chapter II, §4, that the differential $d\pi$ is a linear mapping of $\mathfrak{g}$ onto $(G/H)_o$ and has kernel $\mathfrak{h}$. Let $\mathfrak{m}$ be any subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum). Then $d\pi$ induces an isomorphism of $\mathfrak{m}$ onto $(G/H)_o$. Let $X \in \mathfrak{m}$. Then

$$\mathrm{Ad}_G(h)\, X = dR_{h^{-1}} \circ dL_h(X).$$

Since $\pi \circ R_h = \pi$, $(h \in H)$ and $\pi \circ L_g = \tau(g) \circ \pi$, $(g \in G)$, we obtain

$$(7) \qquad d\pi \circ \mathrm{Ad}_G(h)\, X = d\tau(h) \circ d\pi(X), \qquad h \in H, \quad X \in \mathfrak{m}.$$

The vector $\mathrm{Ad}_G(h)\, X$ decomposes according to $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$,

$$\mathrm{Ad}_G(h)\, X = X(h)_{\mathfrak{h}} + X(h)_{\mathfrak{m}}.$$

The endomorphism $A_h\colon X \to X(h)_{\mathfrak{m}}$ of $\mathfrak{m}$ satisfies

$$d\pi \circ A_h(X) = d\tau(h) \circ d\pi(X), \qquad X \in \mathfrak{m},$$

so that $\det A_h = \det(d\tau(h))$. On the other hand,

$$\exp \mathrm{Ad}_G(h) tT = h \exp tT\, h^{-1} = \exp \mathrm{Ad}_H(h)\, tT$$

for $t \in R$, $T \in \mathfrak{h}$. Hence $\mathrm{Ad}_G(h)\, T = \mathrm{Ad}_H(h)\, T$ so that

$$\det \mathrm{Ad}_G(h) = \det A_h \det \mathrm{Ad}_H(h),$$

and the lemma is proved.

Proposition 1.8. *Let $m = \dim G/H$. The following conditions are equivalent:*

(i) *G/H has a nonzero G-invariant m-form ω;*
(ii) $\det \mathrm{Ad}_G(h) = \det \mathrm{Ad}_H(h)$ *for $h \in H$.*

If these conditions are satisfied, then G/H has a G-invariant orientation and the G-invariant m-form ω is unique up to a constant factor.

Proof. Let ω be a G-invariant m-form on G/H, $\omega \neq 0$. Then the relation $\tau(h)^*\omega = \omega$ at the point o implies $\det(d\tau(h)) = 1$, so (ii) holds. On the other hand, let $X_1, \dots, X_m$ be a basis of $(G/H)_o$ and let $\omega^1, \dots, \omega^m$ be the linear functions on $(G/H)_o$ determined by $\omega^i(X_j) = \delta_{ij}$. Consider the element $\omega^1 \wedge \cdots \wedge \omega^m$ in the Grassmann algebra of the tangent space $(G/H)_o$. Condition (ii) implies that $\det(d\tau(h)) = 1$ and the element $\omega^1 \wedge \cdots \wedge \omega^m$ is invariant under the linear transformation $d\tau(h)$. It follows that there exists a unique G-invariant m-form ω on G/H such that $\omega_o = \omega^1 \wedge \cdots \wedge \omega^m$. If ω^* is another G-invariant m-form on

G/H, then $\omega^* = f\omega$, where $f \in \mathscr{E}(G/H)$. Owing to the G-invariance, $f =$ constant.

Assuming (i), let $\phi: p \to (x_1(p), \ldots, x_m(p))$ be a system of coordinates on an open connected neighborhood U of $o \in G/H$ on which ω has an expression

$$\omega_U = F(x_1, \ldots, x_m)\, dx_1 \wedge \cdots \wedge dx_m,$$

with $F > 0$. The pair $(\tau(g)U,\ \phi \circ \tau(g^{-1}))$ is a local chart on a connected neighborhood of $g \cdot o \in G/H$. We put $(\phi \circ \tau(g^{-1}))(p) = (y_1(p), \ldots, y_m(p))$ for $p \in \tau(g)U$. Then the mapping $\tau(g): U \to \tau(g)U$ has expression ([DS], Chapter I, §3.1) $(y_1, \ldots, y_m) = (x_1, \ldots, x_m)$. On $\tau(g)U$, ω has an expression

$$\omega_{\tau(g)U} = G(y_1, \ldots, y_m)\, dy_1 \wedge \cdots \wedge dy_m$$

and since $\omega_q = \tau(g)^*\omega_{\tau(g)q}$ we have for $q \in U \cap \tau(g)U$

$$\begin{aligned} \omega_q &= G(y_1(q), \ldots, y_m(q))(dy_1 \wedge \cdots \wedge dy_m)_q \\ &= G(x_1(q), \ldots, x_m(q))(dx_1 \wedge \cdots \wedge dx_m)_q. \end{aligned}$$

Hence $F(x_1(q), \ldots, x_m(q)) = G(x_1(q), \ldots, x_m(q))$ and

$$F(x_1(q), \ldots, x_m(q)) = F(y_1(q), \ldots, y_m(q))\, \frac{\partial(y_1(q), \ldots, y_m(q))}{\partial(x_1(q), \ldots, x_m(q))},$$

which shows that the Jacobian of the mapping $(\phi \circ \tau(g^{-1})) \circ \phi^{-1}$ is >0. Consequently, the collection $(\tau(g)U,\ \phi \circ \tau(g^{-1}))_{g \in G}$ of local charts turns G/H into an oriented manifold and each $\tau(g)$ is orientation preserving.

The G-invariant form ω now gives rise to an integral $\int f\omega$ which is invariant in the sense that

$$\int_{G/H} f\omega = \int_{G/H} (f \circ \tau(g))\omega, \qquad g \in G.$$

However, just as the Riemannian measure did not require orientability, an invariant measure can be constructed on G/H under a condition which is slightly more general than (ii). The projective space $\boldsymbol{P}^2(\boldsymbol{R})$ will, for example, satisfy this condition but it does not satisfy (ii). We recall that a measure μ on G/H is said to be invariant (or more precisely G-invariant) if $\mu(f \circ \tau(g)) = \mu(f)$ for all $g \in G$.

Theorem 1.9. *Let G be a Lie group and H a closed subgroup. The relation*

(8) $$|\det \mathrm{Ad}_G(h)| = |\det \mathrm{Ad}_H(h)|, \qquad h \in H,$$

is a necessary and sufficient condition for the existence of a G-invariant measure >0 on G/H. This measure dg_H is unique (up to a constant factor) and

$$(9) \qquad \int_G f(g)\,dg = \int_{G/H}\left(\int_H f(gh)\,dh\right)dg_H, \qquad f \in C_c(G),$$

if the left-invariant measures dg and dh are suitably normalized.

Formula (9) is illustrated in Fig. 6, where $\pi: G \to G/H$ is the natural mapping.

We begin by proving a simple lemma.

Lemma 1.10. *Let G be a Lie group and H a closed subgroup. Let dh be a left-invariant measure >0 on H and put*

$$\bar{f}(gH) = \int_H f(gh)\,dh, \qquad f \in C_c(G).$$

Then the mapping $f \to \bar{f}$ is a linear mapping of $C_c(G)$ onto $C_c(G/H)$.

Proof. Let $F \in C_c(G/H)$; we have to prove that there exists a function $f \in C_c(G)$ such that $F = \bar{f}$. Let C be a compact subset of G/H outside which F vanishes and let C' be a compact subset of G whose image is C under the natural mapping $\pi: G \to G/H$. Let C_H be a compact subset of H of positive measure and put $\tilde{C} = C' \cdot C_H$. Then $\pi(\tilde{C}) = C$. Select $f_1 \in C_c(G)$ such that $f_1 \geq 0$ on G and $f_1 > 0$ on $\tilde{C}$. Then $\bar{f}_1 > 0$ on C (since C_H has positive measure) and the function

$$f(g) = \begin{cases} f_1(g)\dfrac{F(\pi(g))}{\bar{f}_1(\pi(g))} & \text{if } \pi(g) \in C \\ 0 & \text{if } \pi(g) \notin C \end{cases}$$

belongs to $C_c(G)$ and $\bar{f} = F$.

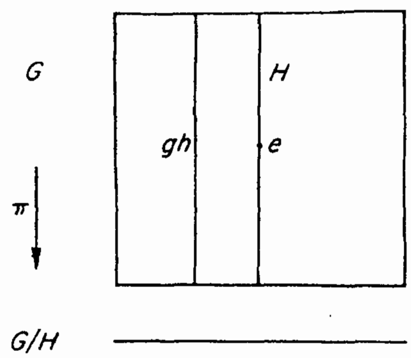

FIG. 6

Now in order to prove Theorem 1.9 suppose first that the relation

$$|\det \mathrm{Ad}_G(h)| = |\det \mathrm{Ad}_H(h)|, \qquad h \in H,$$

holds. Let $\phi \in C_c(G)$. Since we are dealing with measures rather than differential forms, we have by Cor. 1.5

$$\begin{aligned}\int_G \phi(g)\left(\int_H f(gh)\,dh\right)dg &= \int_H dh \int_G \phi(g) f(gh)\,dg \\ &= \int_H dh \int_G \phi(gh^{-1}) f(g) |\det \mathrm{Ad}_G(h)|\,dg \\ &= \int_G f(g)\,dg \int_H \phi(gh^{-1}) |\det \mathrm{Ad}_G(h)|\,dh.\end{aligned}$$

But the relation (8) and the last part of Corollary 1.5 shows that

$$\int_H \phi(gh^{-1}) |\det \mathrm{Ad}_G(h)|\,dh = \int_H \phi(gh)\,dh,$$

so that

$$\int_G \phi(g)\,dg \int_H f(gh)\,dh = \int_G f(g)\,dg \int_H \phi(gh)\,dh.$$

Taking ϕ such that $\int \phi(gh)\,dh = 1$ on the support of f, we conclude that

$$\int_G f(g)\,dg = 0 \qquad \text{if } \tilde{f} \equiv 0.$$

In view of the lemma we can therefore define a linear mapping $\mu: C_c(G/H) \to \boldsymbol{R}$ by

$$\mu(F) = \int_G f(g)\,dg \qquad \text{if} \quad F = \tilde{f}.$$

Since $\mu(F) \geq 0$ if $F \geq 0$, μ is a positive measure on G/H; moreover,

$$\mu((\tilde{f})^{\tau(x)}) = \int_G f^{L(x)}(g)\,dg = \int_G f(g)\,dg = \mu(\tilde{f}),$$

so that μ is invariant.

For the converse we shall first prove the uniqueness of the left Haar measures on G and H.

The uniqueness can be proved as follows. If μ and μ' are two left invariant Haar measures, define ν by $\nu(f) = \mu'(\check{f})$ where $\check{f}(g) = f(g^{-1})$, $f \in C_c(G)$. For $g \in C_c(G)$ consider the function $s \to f(s)\int g(ts)\,d\nu(t)$ and integrate it with respect to μ. Since the ν-integral is constant in s the result is

$$\begin{aligned}\mu(f)\nu(g) &= \int f(s)\int g(ts)\,d\nu(t)\,d\mu(s) = \int d\nu(t)\int f(s)g(ts)\,d\mu(s)\\ &= \int d\nu(t)\int f(t^{-1}\sigma)g(\sigma)\,d\mu(\sigma) = \int g(\sigma)\,d\mu(\sigma)\int f(t^{-1}\sigma)\,d\nu(t).\end{aligned}$$

Put $h(\sigma) = \int f(t^{-1}\sigma)\,d\nu(t)/\mu(f)$. Then the formula shows $\nu = h\mu$ (h independent of f). Taking $\sigma = e$ we deduce

$$h(e)\mu(f) = \nu(\check{f}) = \mu'(f)$$

so μ and μ' are proportional.

If μ is a positive invariant measure on G/H, the mapping $f \to \mu(\tilde{f})$ is a positive left invariant measure on G. Owing to the uniqueness mentioned,

$$\int_G f(g)\,dg = \mu(\tilde{f}).$$

In view of the lemma this proves the uniqueness of μ as well as (9). In order to derive (8), replace $f(g)$ by $f(gh_1)$ in (9). Owing to Corollary 1.5 the left-hand side is multiplied by $|\det \mathrm{Ad}_G(h_1)|$ and the right-hand side is multiplied by $|\det \mathrm{Ad}_H(h_1)|$. This finishes the proof of Theorem 1.9.

Remark. If H is compact, condition (8) is satisfied; hence in this case G/H has a G-invariant measure.

3. Haar Measure in Canonical Coordinates

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Select neighborhoods N_0 of 0 in $\mathfrak{g}$ and N_e of e in G such that the exponential mapping $\exp: \mathfrak{g} \to G$ gives a diffeomorphism of N_0 onto N_e. Fix a Euclidean measure dX on $\mathfrak{g}$ and let dg denote the left-invariant form on G such that $(dg)_e = dX$.

Theorem 1.14. *With dg and dX as above we have for the pullback by* exp

$$(\exp)^*(dg) = \det\left(\frac{1 - e^{-\mathrm{ad}\,X}}{\mathrm{ad}\,X}\right) dX. \tag{12}$$

If $f \in C(G)$ has compact support contained in the canonical coordinate neighborhood N_e, then

$$\int_G f(g)\,dg = \int_{\mathfrak{g}} f(\exp X)\det\left(\frac{1 - e^{-\mathrm{ad}\,X}}{\mathrm{ad}\,X}\right) dX. \tag{13}$$

Proof. Since dg is left-invariant, formula (12) is an immediate consequence of Theorem 1.7, Chapter II, Then (13) follows from (2) in §1 used on the function $f \circ \exp$.

§ 7 Continued

Now let G be a compact connected Lie group. Let dg denote the Haar measure on G normalized by $\int_G dg = 1$, let Q be a fixed positive definite quadratic form on $\mathfrak{g}$ invariant under $\mathrm{Ad}(G)$, and fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ orthonormal with respect to Q. Let $\omega_1, \ldots, \omega_n$ be the left invariant 1-forms on G given by $\omega_i(\tilde{X}_j) = \delta_{ij}$ and put $\theta = \omega_1 \wedge \cdots \wedge \omega_n$. Then θ is left invariant and also right invariant because $\det \mathrm{Ad}(g) \equiv 1$ by the compactness of G. Also each n-form ω on G can be written $\omega = f\theta$ where $f \in C^\infty(G)$ is unique, so we can define

$$\int_G \omega = \int_G f(g)\, dg. \tag{16}$$

Lemma 7.6. *Let ω be an $(n-1)$-form on G. Then*

$$\int_G d\omega = 0.$$

This is a special case of Stokes's theorem and can be proved quickly as follows. We have $d\omega = h\theta$ where $h \in C^\infty(G)$. By (16) and the bi-invariance of dg and θ, we have, since d commutes with mappings and integration with respect to another variable,

$$\int_G d\omega = \int_G \int_G \int_G R(x)^* L(y)^* (d\omega)\, dx\, dy$$

$$= \int_G d\left(\int_{G\times G} R(x)^* L(y)^* \omega\, dx\, dy\right) = 0,$$

the last equality following from Lemma 7.1.

Next we recall the $*$ operator which maps $\mathfrak{A}(G)$ onto itself, $\mathfrak{A}_p(G)$ onto $\mathfrak{A}_{n-p}(G)$ $(0 \leqslant p \leqslant n)$. Let $\sigma_1, \ldots, \sigma_n$ be the basis of the dual space $\mathfrak{g}^*$, dual to (X_i), $\mathfrak{A}(e)$ the Grassmann algebra of $\mathfrak{g} = G_e$, and $\dot{*} : \mathfrak{A}(e) \to \mathfrak{A}(e)$ the mapping determined by linearity and the condition

$$*(\sigma_{i_1} \wedge \ldots \wedge \sigma_{i_p}) = \pm \sigma_{j_1} \wedge \ldots \wedge \sigma_{j_{n-p}}, \tag{17}$$

where $\{i_1, \ldots, i_p, j_1, \ldots, j_{n-p}\}$ is a permutation of $\{1, \ldots, n\}$, the sign being

$+$ or $-$ depending on whether the permutation is even or odd. We shall use the following simple fact from linear algebra (for proofs, see e.g. Flanders [1], Chapter 2):

(i) If (X_i) is replaced by another orthonormal basis (X_j') where $X_j' = \sum_{i=1}^n g_{ij} X_i$ with $\det(g_{ij}) = 1$, then the definition of $*$ does not change.

(ii) If $i_1 < \cdots < i_p$, then

$$\sigma_{i_1} \wedge \ldots \wedge \sigma_{i_p} \wedge *(\sigma_{i_1} \wedge \ldots \wedge \sigma_{i_p}) = \sigma_1 \wedge \ldots \wedge \sigma_n.$$

(iii) $**\sigma = (-1)^{p(n-p)}\sigma$ if $\sigma \in \mathfrak{A}_p(e)$.

From (i) we have since $\det \mathrm{Ad}(g) = 1$, $\mathrm{Ad}(g)* = *\mathrm{Ad}(g)$ $(g \in G)$. Thus we can define $* : \mathfrak{A}(g) \to \mathfrak{A}(g)$ as the map $L(g^{-1})^* * L(g)^*$ or as the map $R(g^{-1})^* * R(g)^*$. Finally, the mapping $* : \mathfrak{A}(G) \to \mathfrak{A}(G)$ is defined by the condition

$$(*\omega)_g = *(\omega_g), \qquad \omega \in \mathfrak{A}(G), \quad g \in G.$$

Then $*$ commutes with $L(x)^*$ and $R(y)^*$ for all $x, y \in G$.

Next we define the linear operator $\delta : \mathfrak{A}(G) \to \mathfrak{A}(G)$ which maps p-forms into $(p-1)$-forms according to the formula

$$\delta\omega = (-1)^{np+n+1} * d * \omega, \qquad \omega \in \mathfrak{A}_p(G).$$

We then introduce an inner product $\langle\ ,\ \rangle$ on $\mathfrak{A}(G)$ by

$$\langle \omega, \eta \rangle = 0 \qquad \text{if} \quad \omega \in \mathfrak{A}_p(G), \quad \eta \in \mathfrak{A}_q(G) \quad (p \neq q),$$

$$\langle \omega, \eta \rangle = \int_G \omega \wedge *\eta \qquad \text{if} \quad \omega, \eta \in \mathfrak{A}_p(G)$$

and the requirement of bilinearity. This inner product is strictly positive definite; in fact we can write

$$\omega = \sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p} \omega_{i_1} \wedge \dots \wedge \omega_{i_p}$$

and then

$$\omega \wedge *\omega = \Big(\sum_{i_1 < \dots < i_p} a^2_{i_1 \dots i_p} \Big) \theta$$

so the statement follows. Moreover d and δ are adjoint operators, that is,

$$\langle d\omega, \eta \rangle = \langle \omega, \delta\eta \rangle, \qquad \omega, \eta \in \mathfrak{A}(G). \tag{18}$$

It suffices to verify this when $\omega \in \mathfrak{A}_{p-1}(G)$, $\eta \in \mathfrak{A}_p(G)$. But then

$$d(\omega \wedge *\eta) = d\omega \wedge *\eta + (-1)^{p-1}\omega \wedge d * \eta = d\omega \wedge *\eta - \omega \wedge *\delta\eta,$$

since $** = (-1)^{p(n-p)}$ on $\mathfrak{A}_p(G)$. Integrating this over G and using Lemma 7.6, we derive (18). We consider now the operator $\Delta = -d\delta - \delta d$ on $\mathfrak{A}(G)$ which maps each $\mathfrak{A}_p(G)$ into itself. A form ω satisfying $\Delta\omega = 0$ is called a *harmonic* form.

Lemma 7.7. *A form ω on G is harmonic if and only if $d\omega = 0$ and $\delta\omega = 0$.*

In fact,

$$-\langle \Delta\omega, \omega \rangle = \langle \delta\omega, \delta\omega \rangle + \langle d\omega, d\omega \rangle$$

so the result follows.

Theorem 7.8. (Hodge) *The harmonic forms on a compact connected Lie group G are precisely the bi-invariant forms.*

A bi-invariant form ω satisfies $d\omega = 0$ (Lemma 7.1); and since $*$ commutes with left and right translations, $\delta\omega = 0$. Conversely, suppose $\Delta\omega = 0$, so by Lemma 7.7, $d\omega = \delta\omega = 0$. Let $X \in \mathfrak{g}$ and let $\tilde{X}$ denote the left invariant vector field on G such that $\tilde{X}_e = X$. By Exercise B.6, Chapter I we have $\theta(\tilde{X})\omega = i(\tilde{X})\, d\omega + di(\tilde{X})\omega = di(\tilde{X})\omega$. Then

$$\langle \theta(\tilde{X})\omega, \theta(\tilde{X})\omega \rangle = \langle \delta\theta(\tilde{X})\omega, i(\tilde{X})\omega \rangle = 0,$$

since $\theta(\tilde{X})\omega$ is harmonic. Hence $\theta(\tilde{X})\omega = 0$, so ω is right invariant (Exercise B.3, Chapter I). Left invariance follows in the same way.

Q.E.D.

§6. Real Forms

Let V be a vector space over $\boldsymbol{R}$ of finite dimension. A *complex structure* on V is an $\boldsymbol{R}$-linear endomorphism J of V such that $J^2 = -I$, where I is the identity mapping of V. A vector space V over $\boldsymbol{R}$ with a complex structure J can be turned into a vector space $\tilde{V}$ over $\boldsymbol{C}$ by putting

$$(a + ib)\,X = aX + bJX,$$

$$X \in V, \quad a, b \in \boldsymbol{R}.$$

In fact, $J^2 = -I$ implies $\alpha(\beta X) = (\alpha\beta)\,X$ for α, $\beta \in \boldsymbol{C}$ and $X \in V$. We have clearly $\dim_{\boldsymbol{C}} \tilde{V} = \frac{1}{2} \dim_{\boldsymbol{R}} V$ and consequently V must be even-dimensional. We call $\tilde{V}$ the *complex vector space associated to* V. Note that V and $\tilde{V}$ agree set theoretically.

On the other hand, if E is a vector space over $\boldsymbol{C}$ we can consider E as a vector space $E^{\boldsymbol{R}}$ over $\boldsymbol{R}$. The multiplication by i on E then becomes a complex structure J on $E^{\boldsymbol{R}}$ and it is clear that $E = (E^{\boldsymbol{R}})^{\sim}$.

A Lie algebra $\mathfrak{v}$ over $\boldsymbol{R}$ is said to have a complex structure J if J is a complex structure on the vector space $\mathfrak{v}$ and in addition

$$[X, JY] = J[X, Y], \qquad \text{for } X, Y \in \mathfrak{v}. \tag{1}$$

Condition (1) means $(\operatorname{ad} X) \circ J = J \circ \operatorname{ad} X$ for all $X \in \mathfrak{v}$, or equivalently, $\operatorname{ad}(JX) = J \circ \operatorname{ad} X$ for all $X \in \mathfrak{v}$. It follows from (1) that

$$[JX, JY] = -[X, Y].$$

The complex vector space $\tilde{\mathfrak{v}}$ becomes a Lie algebra over $\boldsymbol{C}$ with the bracket operation inherited from $\mathfrak{v}$. In fact

$$\begin{aligned}[][(a+ib)\,X, (c+id)\,Y] &= [aX + bJX, cY + dJY] \\ &= ac[X, Y] + bcJ[X, Y] + adJ[X, Y] - bd[X, Y]\end{aligned}$$

so

$$[(a+ib)\,X, (c+id)\,Y] = (a+ib)\,(c+id)\,[X, Y].$$

On the other hand, suppose $\mathfrak{e}$ is a Lie algebra over $\boldsymbol{C}$. The vector space $\mathfrak{e}^R$ has a complex structure J given by multiplication by i on $\mathfrak{e}$. With the bracket operation inherited from $\mathfrak{e}$, $\mathfrak{e}^R$ becomes a Lie algebra over $\boldsymbol{R}$ with the complex structure J.

Now suppose W is an arbitrary finite-dimensional vector space over $\boldsymbol{R}$. The product $W \times W$ is again a vector space over $\boldsymbol{R}$ and the endomorphism $J : (X, Y) \to (-Y, X)$ is a complex structure on $W \times W$. The complex vector space $(W \times W)^{\sim}$ is called the *complexification* of W and will be denoted W^C. We have of course $\dim_C W^C = \dim_R W$. The elements of W^C are the pairs (X, Y) where $X, Y \in W$ and since $(X, Y) = (X, 0) + i(Y, 0)$ we write $X + iY$ instead of (X, Y). Then since

$$(a + bJ)\,(X, Y) = a(X, Y) + b(-Y, X) = (aX - bY, aY + bX)$$

we have

$$(a+ib)\,(X+iY) = aX - bY + i(aY + bX).$$

On the other hand, each finite-dimensional vector space E over $\boldsymbol{C}$ is isomorphic to W^C for a suitable vector space W over $\boldsymbol{R}$; in fact, if (Z_i) is any basis of E, one can take W as the set of all vectors of the form $\sum_i a_i Z_i$, $a_i \in \boldsymbol{R}$.

Let $\mathfrak{l}_0$ be a Lie algebra over $\boldsymbol{R}$; owing to the conventions above, the complex vector space $\mathfrak{l} = (\mathfrak{l}_0)^C$ consists of all symbols $X + iY$, where $X, Y \in \mathfrak{l}_0$. We define the bracket operation in $\mathfrak{l}$ by

$$[X + iY, Z + iT] = [X, Z] - [Y, T] + i([Y, Z] + [X, T]),$$

and this bracket operation is bilinear over C. It is clear that $\mathfrak{l} = (\mathfrak{l}_0)^C$ is a Lie algebra over C; it is called the *complexification of the Lie algebra* $\mathfrak{l}_0$. The Lie algebra $\mathfrak{l}^R$ is a Lie algebra over R with a complex structure J derived from multiplication by i on $\mathfrak{l}$.

Lemma 6.1. *Let K_0, K, and K^R denote the Killing forms of the Lie algebras $\mathfrak{l}_0$, $\mathfrak{l}$, and $\mathfrak{l}^R$. Then*

$$K_0(X, Y) = K(X, Y) \qquad \textit{for } X, Y \in \mathfrak{l}_0,$$

$$K^R(X, Y) = 2\,\mathrm{Re}\,(K(X, Y)) \qquad \textit{for } X, Y \in \mathfrak{l}^R \qquad (\mathrm{Re} = \textit{real part}).$$

The first relation is obvious. For the second, suppose X_i $(1 \leqslant i \leqslant n)$ is any basis of $\mathfrak{l}$; let $B + iC$ denote the matrix of ad X ad Y with respect to this basis, B and C being real. Then $X_1, \ldots, X_n, JX_1, \ldots, JX_n$ is a basis of $\mathfrak{l}^R$ and since the linear transformation ad X ad Y of $\mathfrak{l}^R$ commutes with J, it has the matrix expression

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

and the second relation above follows.

As a consequence of Lemma 6.1 we note that the algebras $\mathfrak{l}_0$, $\mathfrak{l}$, and $\mathfrak{l}^R$ are all semisimple if and only if one of them is.

Definition. Let $\mathfrak{g}$ be a Lie algebra over C. A *real form* of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0$ of the real Lie algebra $\mathfrak{g}^R$ such that

$$\mathfrak{g}^R = \mathfrak{g}_0 + J\mathfrak{g}_0 \qquad \text{(direct sum of vector spaces).}$$

In this case, each $Z \in \mathfrak{g}$ can be uniquely written

$$Z = X + iY, \qquad X, Y \in \mathfrak{g}_0.$$

Thus $\mathfrak{g}$ is isomorphic to the complexification of $\mathfrak{g}_0$. The mapping σ of $\mathfrak{g}$ onto itself given by $\sigma : X + iY \to X - iY$ $(X, Y \in \mathfrak{g}_0)$ is called the *conjugation* of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. The mapping σ has the properties

$$\sigma(\sigma(X)) = X, \qquad \sigma(X + Y) = \sigma(X) + \sigma(Y),$$

$$\sigma(\alpha X) = \bar{\alpha}\sigma(X), \qquad \sigma[X, Y] = [\sigma X, \sigma Y],$$

for $X, Y \in \mathfrak{g}$, $\alpha \in C$. Thus σ is not an automorphism of $\mathfrak{g}$, but it is an automorphism of the real algebra $\mathfrak{g}^R$. On the other hand, let σ be a mapping of $\mathfrak{g}$ onto itself with the properties above. Then the set $\mathfrak{g}_0$ of fixed points of σ is a real form of $\mathfrak{g}$ and σ is the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. In fact, $J\mathfrak{g}_0$ is the eigenspace of σ for the eigenvalue -1 and consequently $\mathfrak{g}^R = \mathfrak{g}_0 + J\mathfrak{g}_0$. If B is the Killing form on $\mathfrak{g} \times \mathfrak{g}$, it is easy to see from Lemma 6.1 that $B(\sigma X, \sigma Y)$ is the complex conjugate of $B(X, Y)$. Another useful remark in this connection is the following: Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two real forms of $\mathfrak{g}$ and σ_1 and σ_2 the corresponding conjugations. Then σ_1 leaves $\mathfrak{g}_2$ invariant if and only if σ_1 and σ_2 commute; in this case we have the direct decompositions

$$\mathfrak{g}_1 = \mathfrak{g}_1 \cap \mathfrak{g}_2 + \mathfrak{g}_1 \cap (i\mathfrak{g}_2),$$
$$\mathfrak{g}_2 = \mathfrak{g}_1 \cap \mathfrak{g}_2 + \mathfrak{g}_2 \cap (i\mathfrak{g}_1).$$

Lemma 6.2. *Suppose $\mathfrak{g}$ is a semisimple Lie algebra over C, $\mathfrak{g}_0$ a real form of $\mathfrak{g}$, and σ the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Let ad denote the adjoint representation of $\mathfrak{g}^R$ and Int $(\mathfrak{g}^R)$ the adjoint group of $\mathfrak{g}^R$. If G_0 denotes the analytic subgroup of Int $(\mathfrak{g}^R)$ whose Lie algebra is ad $(\mathfrak{g}_0)$, then G_0 is a closed subgroup of Int $(\mathfrak{g}^R)$ and analytically isomorphic to Int $(\mathfrak{g}_0)$.*

Proof. Every automorphism s of $\mathfrak{g}^R$ gives rise to an automorphism $\tilde{s}$ of Int $(\mathfrak{g}^R)$ satisfying $\tilde{s}(e^{\operatorname{ad} X}) = e^{\operatorname{ad}(s \cdot X)}$ $(X \in \mathfrak{g}^R)$. In particular there exists an automorphism $\tilde{\sigma}$ of Int $(\mathfrak{g}^R)$ such that $(d\tilde{\sigma})_e (\operatorname{ad} X) = \operatorname{ad}(\sigma \cdot X)$ for $X \in \mathfrak{g}^R$. Since ad is an isomorphism, this proves that ad $(\mathfrak{g}_0)$ is the set of fixed points of $(d\tilde{\sigma})_e$; thus G_0 is the identity component of the set of fixed points of $\tilde{\sigma}$. Now, let ad_0 denote the adjoint representation of $\mathfrak{g}_0$ and for each endomorphism A of $\mathfrak{g}^R$ leaving $\mathfrak{g}_0$ invariant, let A_0 denote its restriction to $\mathfrak{g}_0$. Then if $X \in \mathfrak{g}_0$, we have $(\operatorname{ad} X)_0 = \operatorname{ad}_0 X$ and the mapping $A \to A_0$ maps G_0 onto Int $(\mathfrak{g}_0)$. This mapping is an isomorphism of G_0 onto Int $(\mathfrak{g}_0)$. In fact, suppose $A \in G_0$ such that A_0 is the identity. Since A commutes with the complex structure J, it follows that A is the identity. Finally since the isomorphism is regular at the identity it is an analytic isomorphism.

The following theorem is of fundamental importance in the theory of semisimple Lie algebras and symmetric spaces.

Theorem 6.3. *Every semisimple Lie algebra $\mathfrak{g}$ over C has a real form which is compact.*

The existence of a compact real form was already established by Cartan in 1914 as a biproduct of his classification of real simple Lie algebras. Later when global Lie groups had come to the fore Cartan suggested (without success) the following method for proving the existence of a compact real form. Let $\mathcal{F}$ be the set of all bases $(e_1, \dots, e_n)$ of $\mathfrak{g}$ such that $B(Z, Z) = -\sum_1^n z_i^2$ if $Z = \sum_1^n z_i e_i$ and let c_{ij}^k be the corresponding structure constants. Let f denote the function on $\mathcal{F}$ defined by

$$f(e_1, \dots, e_n) = \sum_{i,j,k} |c_{ij}^k|^2 .$$

Then it is not hard to prove that $\mathfrak{u} = \sum_1^n \mathbf{R} e_i$ is a compact real form of $\mathfrak{g}$ if and only if f has a minimum which is reached for c_{ij}^k real. A proof of the existence of $\mathfrak{u}$ along these lines was accomplished by Richardson (Compact real forms of a complex semisimple Lie algebra, *J. Differential Geometry* 2 (1968), 411-420). In one of the exercises we shall see an important application of Theorem 6.3 in representation theory.

The Classical Groups and Their Cartan Involutions

1. Some Matrix Groups and Their Lie Algebras

In order to describe the real and complex classical groups, we adopt the following (mostly standard) notation. Let $(x_1, ..., x_n)$ and $(z_1, ..., z_n)$ be variable points in $\boldsymbol{R}^n$ and $\boldsymbol{C}^n$, respectively. A matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ operates on $\boldsymbol{C}^n$ by the rule

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \to \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}.$$

As before, E_{ij} denotes the matrix $(\delta_{ai}\delta_{bj})_{1 \leqslant a,b \leqslant n}$. The transpose and conjugate of a matrix A are denoted by ${}^tA$ and $\bar{A}$, respectively; A is called skew symmetric if $A + {}^tA = 0$, Hermitian if ${}^tA = \bar{A}$, skew Hermitian if ${}^tA + \bar{A} = 0$.

If I_n denotes the unit matrix of order n, we put

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \qquad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

$$K_{p,q} = \begin{pmatrix} -I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ 0 & 0 & -I_p & 0 \\ 0 & 0 & 0 & I_q \end{pmatrix}.$$

The multiplicative group of complex numbers of modulus 1 will be denoted by $\boldsymbol{T}$.

$\boldsymbol{GL}(n, \boldsymbol{C})$, $(\boldsymbol{GL}(n, \boldsymbol{R}))$: The group of complex (real) $n \times n$ matrices of determinant $\neq 0$.

$\boldsymbol{SL}(n, \boldsymbol{C})$, $(\boldsymbol{SL}(n, \boldsymbol{R}))$: The group of complex (real) $n \times n$ matrices of determinant 1.

$\boldsymbol{U}(p, q)$: The group of matrices g in $\boldsymbol{GL}(p + q, \boldsymbol{C})$ which leave invariant the Hermitian form

$$-z_1\bar{z}_1 - ... - z_p\bar{z}_p + z_{p+1}\bar{z}_{p+1} + ... + z_{p+q}\bar{z}_{p+q}, \qquad \text{i.e.,} \quad {}^tgI_{p,q}\bar{g} = I_{p,q}.$$

We put $\boldsymbol{U}(n) = \boldsymbol{U}(0, n) = \boldsymbol{U}(n, 0)$ and $\boldsymbol{SU}(p, q) = \boldsymbol{U}(p, q) \cap \boldsymbol{SL}(p + q, \boldsymbol{C})$, $\boldsymbol{SU}(n) = \boldsymbol{U}(n) \cap \boldsymbol{SL}(n, \boldsymbol{C})$. Moreover, let $\boldsymbol{S}(\boldsymbol{U}_p \times \boldsymbol{U}_q)$ denote the set of matrices

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

where $g_1 \in \boldsymbol{U}(p)$, $g_2 \in \boldsymbol{U}(q)$ and $\det g_1 \det g_2 = 1$.

$SU^*(2n)$: The group of matrices in $SL(2n, C)$ which commute with the transformation ψ of C^{2n} given by

$$(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}) \to (\bar{z}_{n+1}, \ldots, \bar{z}_{2n}, -\bar{z}_1, \ldots, -\bar{z}_n).$$

$SO(n, C)$: The group of matrices g in $SL(n, C)$ which leave invariant the quadratic form

$$z_1^2 + \ldots + z_n^2, \qquad \text{i.e.,} \quad {}^t gg = I_n.$$

$SO(p, q)$: The group of matrices g in $SL(p + q, R)$ which leave invariant the quadratic form

$$-x_1^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_{p+q}^2, \qquad \text{i.e.,} \quad {}^t g I_{p,q} g = I_{p,q}.$$

We put $SO(n) = SO(0, n) = SO(n, 0)$.

$SO^*(2n)$: The group of matrices in $SO(2n, C)$ which leave invariant the skew Hermitian form

$$-z_1\bar{z}_{n+1} + z_{n+1}\bar{z}_1 - z_2\bar{z}_{n+2} + z_{n+2}\bar{z}_2 - \ldots - z_n\bar{z}_{2n} + z_{2n}\bar{z}_n.$$

Thus $g \in SO^*(2n) \Leftrightarrow {}^t g J_n \bar{g} = J_n, {}^t gg = I_{2n}$.

$Sp(n, C)$: The group of matrices g in $GL(2n, C)$ which leave invariant the exterior form

$$z_1 \wedge z_{n+1} + z_2 \wedge z_{n+2} + \ldots + z_n \wedge z_{2n}, \qquad \text{i.e.,} \quad {}^t g J_n g = J_n.$$

$Sp(n, R)$: The group of matrices g in $GL(2n, R)$ which leave invariant the exterior form

$$x_1 \wedge x_{n+1} + x_2 \wedge x_{n+2} + \ldots + x_n \wedge x_{2n}, \qquad \text{i.e.,} \quad {}^t g J_n g = J_n.$$

$Sp(p, q)$: The group of matrices g in $Sp(p + q, C)$ which leave invariant the Hermitian form

$${}^t Z K_{p,q} \bar{Z}, \qquad \text{i.e.,} \quad {}^t g K_{p,q} \bar{g} = K_{p,q}.$$

We put $Sp(n) = Sp(0, n) = Sp(n, 0)$. It is clear that $Sp(n) = Sp(n, C) \cap U(2n)$.

The groups listed above are all topological Lie subgroups of a general linear group. The Lie algebra of the general linear group $GL(n, C)$ can (as in Chapter II, §1) be identified with the Lie algebra $\mathfrak{gl}(n, C)$ of all complex $n \times n$ matrices, the bracket operation being $[A, B] = AB - BA$. The Lie algebra for each of the groups above is then canonically identified with a subalgebra of $\mathfrak{gl}(n, C)$, considered as a real Lie algebra. These Lie algebras will be denoted by the corresponding small German letters, $\mathfrak{sl}(n, R)$, $\mathfrak{su}(p, q)$, etc.

Now, if G is a Lie group with Lie algebra $\mathfrak{g}$, then the Lie algebra $\mathfrak{h}$ of a topological Lie subgroup H of G is given by

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp tX \in H \text{ for } t \in \boldsymbol{R}\}. \tag{1}$$

Using this fact (Chapter II, §2) we can describe the Lie algebras of the groups above more explicitly. Since the computation is fairly similar for all the groups we shall give the details only in the cases $\boldsymbol{SU^*(2n)}$ and $\boldsymbol{Sp(n, C)}$. Case $\boldsymbol{SO(p, q)}$ was done in Chapter V, §2.

$\mathfrak{gl}(n, \boldsymbol{C})$, $(\mathfrak{gl}(n, \boldsymbol{R}))$: {all $n \times n$ complex (real) matrices},

$\mathfrak{sl}(n, \boldsymbol{C})$, $(\mathfrak{sl}(n, \boldsymbol{R}))$: {all $n \times n$ complex (real) matrices of trace 0},

$$\mathfrak{u}(p, q) : \left\{\begin{pmatrix} Z_1 & Z_2 \\ {}^t\bar{Z}_2 & Z_3 \end{pmatrix} \,\middle|\, \begin{matrix} Z_1, Z_3 \text{ skew Hermitian of order } p \text{ and } q, \\ \text{respectively, } Z_2 \text{ arbitrary} \end{matrix}\right\},$$

$$\mathfrak{su}(p, q) : \left\{\begin{pmatrix} Z_1 & Z_2 \\ {}^t\bar{Z}_2 & Z_3 \end{pmatrix} \,\middle|\, \begin{matrix} Z_1, Z_3 \text{ skew Hermitian, of order } p \text{ and } q, \\ \text{respectively, } \operatorname{Tr} Z_1 + \operatorname{Tr} Z_3 = 0,\ Z_2 \text{ arbitrary} \end{matrix}\right\},$$

$$\mathfrak{su}^*(2n) : \left\{\begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \,\middle|\, \begin{matrix} Z_1, Z_2\ n \times n \text{ complex matrices} \\ \operatorname{Tr} Z_1 + \operatorname{Tr} \bar{Z}_1 = 0 \end{matrix}\right\},$$

$\mathfrak{so}(n, \boldsymbol{C})$: {all $n \times n$ skew symmetric complex matrices},

$$\mathfrak{so}(p, q) : \left\{\begin{pmatrix} X_1 & X_2 \\ {}^tX_2 & X_3 \end{pmatrix} \,\middle|\, \begin{matrix} \text{All } X_i \text{ real, } X_1, X_3 \text{ skew symmetric of order} \\ p \text{ and } q, \text{ respectively, } X_2 \text{ arbitrary} \end{matrix}\right\},$$

$$\mathfrak{so}^*(2n) : \left\{\begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \,\middle|\, \begin{matrix} Z_1, Z_2\ n \times n \text{ complex matrices,} \\ Z_1 \text{ skew, } Z_2 \text{ Hermitian} \end{matrix}\right\},$$

$$\mathfrak{sp}(n, \boldsymbol{C}) : \left\{\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -{}^tZ_1 \end{pmatrix} \,\middle|\, \begin{matrix} Z_i \text{ complex } n \times n \text{ matrices,} \\ Z_2 \text{ and } Z_3 \text{ symmetric} \end{matrix}\right\},$$

$$\mathfrak{sp}(n, \boldsymbol{R}) : \left\{\begin{pmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{pmatrix} \,\middle|\, \begin{matrix} X_1, X_2, X_3 \text{ real } n \times n \text{ matrices,} \\ X_2, X_3 \text{ symmetric} \end{matrix}\right\},$$

$$\mathfrak{sp}(p, q) : \left\{\begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ {}^t\bar{Z}_{12} & Z_{22} & {}^tZ_{14} & Z_{24} \\ -\bar{Z}_{13} & \bar{Z}_{14} & \bar{Z}_{11} & -\bar{Z}_{12} \\ {}^t\bar{Z}_{14} & -\bar{Z}_{24} & -{}^tZ_{12} & \bar{Z}_{22} \end{pmatrix} \,\middle|\, \begin{matrix} Z_{ij} \text{ complex matrix; } Z_{11} \text{ and } Z_{13} \text{ of} \\ \text{order } p,\ Z_{12} \text{ and } Z_{14}\ p \times q \text{ matrices,} \\ Z_{11} \text{ and } Z_{22} \text{ are skew Hermitian,} \\ Z_{13} \text{ and } Z_{24} \text{ are symmetric} \end{matrix}\right\}.$$

Proof for $\boldsymbol{SU^*(2n)}$. By the definition of this group, we have $g \in \boldsymbol{SU^*(2n)}$ if and only if $g\psi = \psi g$ and $\det g = 1$. This shows that $A \in \mathfrak{su}^*(2n)$ if and only if $A\psi = \psi A$ and $\operatorname{Tr} A = 0$. Writing A in the form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where A_i are $n \times n$ complex matrices we see that if U and V are $n \times 1$ matrices, then

$$A\psi\begin{pmatrix}U\\V\end{pmatrix} = \begin{pmatrix}A_1 & A_2\\A_3 & A_4\end{pmatrix}\begin{pmatrix}\bar{V}\\-\bar{U}\end{pmatrix} = \begin{pmatrix}A_1\bar{V} - A_2\bar{U}\\A_3\bar{V} - A_4\bar{U}\end{pmatrix},$$

$$\psi A\begin{pmatrix}U\\V\end{pmatrix} = \psi\begin{pmatrix}A_1U + A_2V\\A_3U + A_4V\end{pmatrix} = \begin{pmatrix}\bar{A}_3\bar{U} + \bar{A}_4\bar{V}\\-\bar{A}_1\bar{U} - \bar{A}_2\bar{V}\end{pmatrix}.$$

It follows that $\bar{A}_3 = -A_2$, $A_1 = \bar{A}_4$ as desired.

Proof for $Sp(n, C)$. Writing symbolically

$$2(z_1 \wedge z_{n+1} + \dots + z_n \wedge z_{2n}) = (z_1, \dots, z_{2n}) \wedge J_n{}^t(z_1, \dots, z_{2n})$$

it is clear that $g \in Sp(n, C)$ if and only if

$$^t g J_n g = J_n.$$

Using this for $g = \exp tZ$ $(t \in R)$, we find since $A \exp ZA^{-1} = \exp(AZA^{-1})$, $^t(\exp Z) = \exp {}^tZ$,

$$\exp t(J_n^{-1}\,{}^tZJ_n) = \exp(-tZ) \qquad (t \in R),$$

so $Z \in \mathfrak{sp}(n, C)$ if an only if

$$^tZJ_n + J_nZ = 0. \tag{2}$$

Writing Z in the form

$$Z = \begin{pmatrix}Z_1 & Z_2\\Z_3 & Z_4\end{pmatrix},$$

where Z_i is a complex $n \times n$ matrix, condition (2) is equivalent to $^tZ_1 + Z_4 = 0$, $Z_2 = {}^tZ_2$, $Z_3 = {}^tZ_3$.

2. Connectivity Properties

Having described the Lie algebras, we shall now discuss the connectivity of the groups defined.

Lemma 2.1. *Let $\approx$ denote topological isomorphism, and $\sim$ a homeomorphism. We then have*

(a) $SO(2n) \cap Sp(n) \approx U(n)$.

(b) $Sp(p, q) \cap U(2p + 2q) \approx Sp(p) \times Sp(q)$.

(c) $Sp(n, R) \cap U(2n) \approx U(n)$.

(d) $SO^*(2n) \cap U(2n) \approx U(n)$.

(e) $SU(p, q) \cap U(p+q) = S(U_p \times U_q) \sim SU(p) \times T \times SU(q)$.

(f) $SU^*(2n) \cap U(2n) = Sp(n)$.

Proof. (a) Each $g \in Sp(n)$ has determinant 1 so $g \in SO(2n) \cap Sp(n)$ is equivalent to ${}^t gg = I_{2n}$, ${}^t g J_n g = J_n$, ${}^t g\bar{g} = I_{2n}$. Writing

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

these last relations amount to g real, $A = D$, $B = -C$, $A{}^tB - B{}^tA = 0$, $A{}^tA + B{}^tB = I_n$. But the last two formulas express simply $A + iB \in U(n)$. For part (b), let

$$V = \{g \in GL(2p+2q, C) : {}^t g K_{p,q} \bar{g} = K_{p,q}\}.$$

Then

$$g \in U(2p+2q) \cap V \quad \Leftrightarrow \quad {}^t g\bar{g} = I_{2p+2q}, \qquad {}^t g K_{p,q} \bar{g} = K_{p,q}.$$

But the last two relations are equivalent to

$$g = \begin{pmatrix} X_{11} & 0 & X_{13} & 0 \\ 0 & X_{22} & 0 & X_{24} \\ X_{31} & 0 & X_{33} & 0 \\ 0 & X_{42} & 0 & X_{44} \end{pmatrix} \qquad \text{where} \qquad \begin{array}{l} \begin{pmatrix} X_{11} & X_{13} \\ X_{31} & X_{33} \end{pmatrix} \in U(2p) \\ \begin{pmatrix} X_{22} & X_{24} \\ X_{42} & X_{44} \end{pmatrix} \in U(2q). \end{array} \tag{3}$$

By definition

$$Sp(p, q) = Sp(p+q, C) \cap V$$

so

$$Sp(p, q) \cap U(2p+2q) = Sp(p+q, C) \cap U(2p+2q) \cap V.$$

Thus, g in (3) belongs to $Sp(p, q) \cap U(2p+2q)$ if and only if ${}^t g J_{p+q} g = J_{p+q}$ or equivalently

$$\begin{pmatrix} X_{11} & X_{13} \\ X_{31} & X_{33} \end{pmatrix} \in U(2p) \cap Sp(p, C) = Sp(p)$$

and

$$\begin{pmatrix} X_{22} & X_{24} \\ X_{42} & X_{44} \end{pmatrix} \in U(2q) \cap Sp(q, C) = Sp(q).$$

This proves (b). For (c) we only have to note that

$$Sp(n, R) \cap U(2n) = Sp(n) \cap SO(2n),$$

which by (a) is isomorphic to $U(n)$. Part (d) is also easy; in fact, $g \in SO^*(2n)$ by definition if and only if ${}^t g g = I_{2n}$ and ${}^t g J_n \bar{g} = J_n$.

Thus

$$SO^*(2n) \cap U(2n) = SO(2n) \cap Sp(n, C) = SO(2n) \cap Sp(n) \approx U(n).$$

Part (e). We have

$$g \in SU(p, q) \cap U(p+q) \quad \Leftrightarrow \quad g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

where $g_1 \in U(p)$, $g_2 \in U(q)$ and $\det g_1 \det g_2 = 1$. Such a matrix can be written

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} = \begin{pmatrix} \det g_1 & 0 & 0 & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ \vdots & & & \ddots \\ 0 & & 1 & 0 \\ 0 & & 0 & \det g_2 \end{pmatrix} \cdot \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix},$$

where $\gamma_1 \in SU(p)$, $\gamma_2 \in SU(q)$. We have therefore a mapping

$$g \to (\gamma_1, \det g_1, \gamma_2)$$

of $SU(p, q) \cap U(p+q)$ into $SU(p) \times T \times SU(q)$. This mapping is not in general a homomorphism but it is continuous, one-to-one and onto; hence $SU(p, q) \cap U(p+q)$ is homeomorphic to $SU(p) \times T \times SU(q)$. Finally, $g \in SU^*(2n)$ if and only if $\bar{g} J_n = J_n g$ and $\det g = 1$. Hence $g \in SU^*(2n) \cap U(2n)$ if and only if $\bar{g} J_n = J_n g$, ${}^t g \bar{g} = I_{2n}$, $\det g = 1$. However, these conditions are equivalent to ${}^t g J_n g = J_n$, ${}^t g \bar{g} = I_{2n}$ or $g \in Sp(n)$. This finishes the proof of the lemma.

The following lemma is well known, see, e.g., Chevalley [**Lie Groups I**,

Lemma 2.2.

(a) *The groups* $GL(n, C)$, $SL(n, C)$, $SL(n, R)$, $SO(n, C)$, $SO(n)$, $SU(n)$, $U(n)$, $Sp(n, C)$, $Sp(n)$ *are all connected.*

(b) *The group* $GL(n, R)$ *has two connected components.*

In order to determine the connectivity of the remaining groups we need another lemma.

Definition. Let G be a subgroup of the general linear group $GL(n, C)$. Let $z_{ij}(\sigma)$ $(1 \leqslant i, j \leqslant n)$ denote the matrix elements of an arbitrary $\sigma \in GL(n, C)$, and let $x_{ij}(\sigma)$ and $y_{ij}(\sigma)$ be the real and imaginary part of $z_{ij}(\sigma)$. The group G is called a *pseudoalgebraic* subgroup of $GL(n, C)$

if there exists a set of polynomials P_β in $2n^2$ arguments such that $\sigma \in G$ if and only if $P_\beta(\dots x_{ij}(\sigma), y_{ij}(\sigma), \dots) = 0$ for all P_β.

A pseudoalgebraic subgroup of $GL(n, C)$ is a closed subgroup, hence a topological Lie subgroup.

Lemma 2.3. *Let G be a pseudoalgebraic subgroup of $GL(n, C)$ such that the condition $g \in G$ implies ${}^t\bar{g} \in G$. Then there exists an integer $d \geqslant 0$ such that G is homeomorphic to the topological product of $G \cap U(n)$ and R^d.*

Proof. We first remark that if an exponential polynomial $Q(t) = \sum_{j=1}^{n} c_j e^{b_j t}$ $(b_j \in R,\ c_j \in C)$ vanishes whenever t is an integer then $Q(t) = 0$ for all $t \in R$. Let $\mathfrak{h}(n)$ denote the vector space of all Hermitian $n \times n$ matrices. Then exp maps $\mathfrak{h}(n)$ homeomorphically onto the space $P(n)$ of all positive definite Hermitian $n \times n$ matrices (see Chevalley **Theory of Lie Groups**, Chapter I). Let $H \in \mathfrak{h}(n)$. We shall prove

$$\text{If } \exp H \in G \cap P(n), \text{ then } \exp tH \in G \cap P(n) \text{ for } t \in R. \tag{4}$$

There exists a matrix $u \in U(n)$ such that uHu^{-1} is a diagonal matrix. Since the group uGu^{-1} is pseudoalgebraic as well as G, we may assume that H in (4) is a diagonal matrix. Let $h_1, \dots, h_n$ be the (real) diagonal elements of H. The condition $\exp H \in G \cap P(n)$ means that the numbers $e^{h_1}, \dots, e^{h_n}$ satisfy a certain set of algebraic equations. Since $\exp kH \in G \cap P(n)$ for each integer k, the numbers $e^{kh_1}, \dots, e^{kh_n}$ also satisfy these algebraic equations and by the remark above the same is the case if k is any real number. This proves (4).

Each $g \in GL(n, C)$ can be decomposed uniquely $g = up$ where $u \in U(n)$, $p \in P(n)$. Here u and p depend continuously on g. If $g \in G$, then ${}^t\bar{g}g = p^2 \in G \cap P(n)$, so by (4) $p \in G \cap P(n)$ and $u \in G \cap U(n)$. The mapping $g \to (u, p)$ is a one-to-one mapping of G onto the product $(G \cap U(n)) \times (G \cap P(n))$ and since G carries the relative topology of $GL(n, C)$, this mapping is a homeomorphism.

The Lie algebra $\mathfrak{gl}(n, C)$ is a direct sum

$$\mathfrak{gl}(n, C) = \mathfrak{u}(n) + \mathfrak{h}(n).$$

Since the Lie algebra $\mathfrak{g}$ of G is invariant under the involutive automorphism $X \to -{}^t\bar{X}$ of $\mathfrak{gl}(n, C)$ we have

$$\mathfrak{g} = \mathfrak{g} \cap \mathfrak{u}(n) + \mathfrak{g} \cap \mathfrak{h}(n).$$

It is obvious that $\exp(\mathfrak{g} \cap \mathfrak{h}(n)) \subset G \cap P(n)$. On the other hand, each $p \in G \cap P(n)$ can be written uniquely $p = \exp H$ where $H \in \mathfrak{h}(n)$; by

(4), $H \in \mathfrak{h}(n) \cap \mathfrak{g}$, so exp induces a homeomorphism of $\mathfrak{g} \cap \mathfrak{h}(n)$ onto $G \cap P(n)$. This proves the lemma.

Lemma 2.4.

(a) *The groups $SU(p, q)$, $SU^*(2n)$, $SO^*(2n)$, $Sp(n, R)$, and $Sp(p, q)$ are all connected.*

(b) *The group $SO(p, q)$ $(0 < p < p + q)$ has two connected components.*

Proof. All these groups are pseudoalgebraic subgroups of the corresponding general linear group and have the property that $g \in G \Rightarrow {}^t\bar{g} \in G$. Part (a) is therefore an immediate consequence of Lemma 2.3 and Lemma 2.1. For (b) we consider the intersection $SO(p, q) \cap U(p + q) = SO(p, q) \cap SO(p + q)$. This consists of all matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where A and B are orthogonal matrices of order p and q respectively satisfying $\det A \det B = 1$. It follows again from Lemma 2.3 that $SO(p, q)$ has two components.

3. The Involutive Automorphisms of the Classical Compact Lie Algebras

Let $\mathfrak{u}$ be a compact simple Lie algebra, θ an involutive automorphism of $\mathfrak{u}$; let $\mathfrak{u} = \mathfrak{k}_0 + \mathfrak{p}_*$ be the decomposition of $\mathfrak{u}$ into eigenspaces of θ and let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ (where $\mathfrak{p}_0 = i\mathfrak{p}_*$). Then $\mathfrak{g}_0$ is a real form of the complexification $\mathfrak{g} = \mathfrak{u}^C$. We list below the "classical" $\mathfrak{u}$, that is, $\mathfrak{su}(n)$, $\mathfrak{so}(n)$, and $\mathfrak{sp}(n)$ and for each give various θ; later these will be shown to exhaust all possibilities for θ up to conjugacy. Then $\mathfrak{g}_0$ runs through all noncompact real forms of $\mathfrak{g}$ up to isomorphism. The simply connected Riemannian globally symmetric spaces corresponding to $(\mathfrak{u}, \theta)$ and $\mathfrak{g}_0$ are also listed (for $\mathfrak{u}$ classical). As earlier, $\mathfrak{h}_{\mathfrak{p}_*}$ and $\mathfrak{h}_{\mathfrak{p}_0}$ denote maximal abelian subspaces of $\mathfrak{p}_*$ and $\mathfrak{p}_0$, respectively.

Type A I $\quad \mathfrak{u} = \mathfrak{su}(n)$; $\theta(X) = \bar{X}$.

Here $\mathfrak{k}_0 = \mathfrak{so}(n)$ and $\mathfrak{p}_*$ consists of all symmetric purely imaginary $n \times n$ matrices of trace 0. Thus $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 = \mathfrak{sl}(n, R)$. The corresponding simply connected symmetric spaces are

$$SL(n, R)/SO(n), \qquad SU(n)/SO(n) \qquad (n > 1).$$

The diagonal matrices in $\mathfrak{p}_*$ form a maximal abelian subspace. Hence the *rank* is $n - 1$. Since $\mathfrak{g} = \mathfrak{a}_{n-1}$, the algebra $\mathfrak{g}_0$ is a *normal* real form of $\mathfrak{g}$.

Type A II $\mathfrak{u} = \mathfrak{su}(2n)$; $\theta(X) = J_n \bar{X} J_n^{-1}$.
Here $\mathfrak{k}_0 = \mathfrak{sp}(n)$ and

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ \bar{Z}_2 & -\bar{Z}_1 \end{pmatrix} \middle| Z_1 \in \mathfrak{su}(n), Z_2 = \mathfrak{so}(n, C) \right\}.$$

Hence $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 = \mathfrak{su}^*(2n)$. The corresponding simply connected symmetric spaces are

$$SU^*(2n)/Sp(n), \qquad SU(2n)/Sp(n) \qquad (n > 1).$$

The diagonal matrices in $\mathfrak{p}_*$ form a maximal abelian subspace of $\mathfrak{p}_*$. Hence the *rank* is $n - 1$.

Type A III $\mathfrak{u} = \mathfrak{su}(p + q)$; $\theta(X) = I_{p,q} X I_{p,q}$.
Here

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \begin{matrix} A \in \mathfrak{u}(p), B \in \mathfrak{u}(q) \\ \mathrm{Tr}(A + B) = 0 \end{matrix} \right\},$$

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} 0 & Z \\ -{}^t\bar{Z} & 0 \end{pmatrix} \middle| Z \quad p \times q \text{ complex matrix} \right\}.$$

The decomposition

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$$= \begin{pmatrix} A - \frac{1}{p}(\mathrm{Tr}\, A) I_p & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{p}(\mathrm{Tr}\, A) I_p & 0 \\ 0 & \frac{1}{q}(\mathrm{Tr}\, B) I_q \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & B - \frac{1}{q}(\mathrm{Tr}\, B) I_q \end{pmatrix}$$

shows that $\mathfrak{k}_0$ is isomorphic to the product

$$\mathfrak{su}(p) \times \mathfrak{c}_0 \times \mathfrak{su}(q),$$

where $\mathfrak{c}_0$ is the center of $\mathfrak{k}_0$. Also $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 = \mathfrak{su}(p, q)$. The corresponding simply connected symmetric spaces are

$$SU(p, q)/S(U_p \times U_q), \qquad SU(p + q)/S(U_p \times U_q) \qquad (p \geqslant 1, q \geqslant 1, p \geqslant q).$$

A maximal abelian subspace of $\mathfrak{p}_*$ is given by

$$\mathfrak{h}_{\mathfrak{p}_*} = \sum_{i=1}^{q} R(E_{i\ p+i} - E_{p+i\ i}). \tag{5}$$

Consequently, the *rank* is q. The spaces are *Hermitian symmetric*. For $q = 1$, these spaces are the so-called *Hermitian hyperbolic space* and the *complex projective space*.

Type BD I $\mathfrak{u} = \mathfrak{so}(p+q)$; $\theta(X) = I_{p,q}XI_{p,q}$ $(p \geqslant q)$.
Here

$$\mathfrak{k}_0 = \left\{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \,\middle|\, A \in \mathfrak{so}(p),\ B \in \mathfrak{so}(q)\right\},$$

$$\mathfrak{p}_* = \left\{\begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} \,\middle|\, X \text{ real } p \times q \text{ matrix}\right\}.$$

As shown in Chapter V, §2, the mapping

$$\begin{pmatrix} A & iX \\ -i{}^tX & B \end{pmatrix} \to \begin{pmatrix} A & X \\ {}^tX & B \end{pmatrix}$$

is an isomorphism of $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ onto $\mathfrak{so}(p, q)$. The simply connected symmetric spaces associated with $\mathfrak{so}(p, q)$ and $(\mathfrak{u}, \theta)$ are

$$SO_0(p,q)/SO(p) \times SO(q), \quad SO(p+q)/SO(p) \times SO(q) \quad \begin{pmatrix} p > 1, q \geqslant 1 \\ p+q \neq 4, p \geqslant q \end{pmatrix}.$$

Here $SO_0(p, q)$ denotes the identity component of $SO(p, q)$. The compact space is the manifold of oriented p-planes of $(p+q)$-space, which is known (see, e.g., Steenrod [1], p. 134) to be simply connected. A maximal abelian subspace of $\mathfrak{p}_*$ is again given by (5), so the *rank* is q. If $p+q$ is even then $\mathfrak{g}_0$ is a *normal* real form of $\mathfrak{g}$ if and only if $p = q$. If $p+q$ is odd then $\mathfrak{g}_0$ is a *normal* real form of $\mathfrak{g}$ if and only if $p = q + 1$.

For $q = 1$, the spaces are the *real hyperbolic space* and the *sphere*. These are the simply connected Riemannian manifolds of constant sectional curvature $\neq 0$ and dimension $\neq 3$. Those of dimension 3 are $SL(2, C)$ $SU(2)$ and $SU(2)$, i.e., $\mathfrak{a}_n$ for $n = 1$.

If $q = 2$, then $\mathfrak{k}_0$ has nonzero center and the spaces are *Hermitian symmetric*.

Type D III $\mathfrak{u} = \mathfrak{so}(2n)$; $\theta(X) = J_nXJ_n^{-1}$.
Here $\mathfrak{k}_0 = \mathfrak{so}(2n) \cap \mathfrak{sp}(n)$ which by Lemma 2.1 is isomorphic to $\mathfrak{u}(n)$. Moreover,

$$\mathfrak{p}_* = \left\{\begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \,\middle|\, X_1, X_2 \in \mathfrak{so}(n)\right\}.$$

Hence $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 = \mathfrak{so}^*(2n)$. The symmetric spaces are

$$SO^*(2n)/U(n), \qquad SO(2n)/U(n) \qquad (n > 2).$$

Here the imbedding of $U(n)$ into $SO(2n)$, (and $SO^*(2n)$), is given by the mapping

$$A + iB \to \begin{Bmatrix} A & B \\ -B & A \end{Bmatrix}, \tag{6}$$

where $A + iB \in U(n)$, A, B real. The spaces are *Hermitian symmetric* since $\mathfrak{k}_0$ has nonzero center. In view of Theorem 4.6, Chapter VIII, they are simply connected. A maximal abelian subspace of $\mathfrak{p}_*$ is spanned by the matrices

$$(E_{12} - E_{21}) - (E_{n+1\,n+2} - E_{n+2\,n+1}),\quad (E_{23} - E_{32}) - (E_{n+2\,n+3} - E_{n+3\,n+2}), \dots$$

Consequently, the *rank* is $[n/2]$.

Type C I $\mathfrak{u} = \mathfrak{sp}(n)$; $\theta(X) = \bar{X}\ (= J_n X J_n^{-1})$.
Here $\mathfrak{k}_0 = \mathfrak{sp}(n) \cap \mathfrak{so}(2n)$ which is isomorphic to $\mathfrak{u}(n)$.

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & -Z_1 \end{pmatrix} \middle| \begin{array}{l} Z_1 \in \mathfrak{u}(n), \text{ purely imaginary} \\ Z_2 \text{ symmetric, purely imaginary} \end{array} \right\}.$$

Hence $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 = \mathfrak{sp}(n, R)$. The corresponding simply connected symmetric spaces are

$$Sp(n, R)/U(n), \qquad Sp(n)/U(n) \qquad (n \geqslant 1).$$

Here the imbedding of $U(n)$ into $Sp(n)$ (and $Sp(n, R)$) is given by (6). The diagonal matrices in $\mathfrak{p}_*$ form a maximal abelian subspace. Thus the spaces have *rank* n and $\mathfrak{g}_0$ is a *normal* real form of $\mathfrak{g}$. The spaces are *Hermitian symmetric*.

Type C II $\mathfrak{u} = \mathfrak{sp}(p + q)$; $\theta(X) = K_{p,q} X K_{p,q}$.
Here

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} X_{11} & 0 & X_{13} & 0 \\ 0 & X_{22} & 0 & X_{24} \\ -\bar{X}_{13} & 0 & \bar{X}_{11} & 0 \\ 0 & -\bar{X}_{24} & 0 & \bar{X}_{22} \end{pmatrix} \middle| \begin{array}{l} X_{11} \in \mathfrak{u}(p),\ X_{22} \in \mathfrak{u}(q) \\ X_{13}\ p \times p \text{ symmetric} \\ X_{24}\ q \times q \text{ symmetric} \end{array} \right\},$$

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} 0 & Y_{12} & 0 & Y_{14} \\ -{}^t\bar{Y}_{12} & 0 & {}^tY_{14} & 0 \\ 0 & -\bar{Y}_{14} & 0 & \bar{Y}_{12} \\ -{}^t\bar{Y}_{14} & 0 & -{}^tY_{12} & 0 \end{pmatrix} \middle| \begin{array}{l} Y_{12} \text{ and } Y_{14} \text{ arbitrary} \\ \text{complex } p \times q \text{ matrices} \end{array} \right\}.$$

It is clear that $\mathfrak{k}_0$ is isomorphic to the direct product $\mathfrak{sp}(p) \times \mathfrak{sp}(q)$. Moreover, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 = \mathfrak{sp}(p, q)$. The corresponding simply connected symmetric spaces are

$$Sp(p, q)/Sp(p) \times Sp(q), \qquad Sp(p + q)/Sp(p) \times Sp(q) \qquad (p \geqslant q \geqslant 1).$$

Here the imbedding of $Sp(p) \times Sp(q)$ into $Sp(p+q)$ (and $Sp(p, q)$) is given by the mapping

$$\left(\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\right) \rightarrow \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}$$

A maximal abelian subspace of $\mathfrak{p}_*$ is obtained by taking $Y_{14} = 0$ and letting Y_{12} run through the space $RE_{11} + RE_{22} + \cdots + RE_{qq}$. Consequently, the *rank* is q. For $q = 1$, the spaces are the so-called *quaternian hyperbolic spaces* and the *quaternian projective spaces.*

This will be shown to exhaust all involutive automorphisms of the compact classical simple Lie algebras. The restriction on the indices is made in order that the algebras should be simple, the spaces of dimension >0, and the condition $p \geqslant q$ is required in order to avoid repetition within the same class.