10. Quantization of Claogero-Moser spaces

10.1. **Quantum moment maps and quantum Hamiltonian reduction.** Now we would like to quantize the notion of a moment map. Let $\mathfrak{g}$ be a Lie algebra, and A be an associative algebra equipped with a $\mathfrak{g}$-action, i.e. a Lie algebra map $\phi : \mathfrak{g} \to \mathrm{D}erA$. A quantum moment map for (A, ϕ) is an associative algebra homomorphism $\mu : U(\mathfrak{g}) \to A$ such that for any $a \in \mathfrak{g}$, $b \in A$ one has $[\mu(a), b] = \phi(a)b$.

The space of $\mathfrak{g}$-invariants $A^{\mathfrak{g}}$, i.e. elements $b \in A$ such that $[\mu(a), b] = 0$ for all $a \in \mathfrak{g}$, is a subalgebra of A. Let $J \subset A$ be the left ideal generated by $\mu(a)$, $a \in \mathfrak{g}$. Then J is not a 2-sided ideal, but $J^{\mathfrak{g}} := J \cap A^{\mathfrak{g}}$ is a 2-sided ideal in $A^{\mathfrak{g}}$. Indeed, let $c \in A^{\mathfrak{g}}$, and $b \in J^{\mathfrak{g}}$, $b = \sum_i b_i\mu(a_i)$, $b_i \in A, a_i \in \mathfrak{g}$. Then $bc = \sum b_i\mu(a_i)c = \sum b_ic\mu(a_i) \in J^{\mathfrak{g}}$.

Thus, the algebra $A//\mathfrak{g} := A^{\mathfrak{g}}/J^{\mathfrak{g}}$ is an associative algebra, which is called the quantum Hamiltonian reduction of A with respect to the quantum moment map μ.

10.2. **The Levasseur-Stafford theorem.** In general, similarly to the classical case, it is rather difficult to compute the quantum reduction $A//\mathfrak{g}$. For example, in this subsection we will describe $A//\mathfrak{g}$ in the case when $A = \mathcal{D}(\mathfrak{g})$ is the algebra of differential operators on a reductive Lie algebra $\mathfrak{g}$, and $\mathfrak{g}$ acts on A through the adjoint action on itself. This description is a very nontrivial result of Levasseur and Stafford.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and W the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{h}_{\mathrm{reg}}$ denote the set of regular points in $\mathfrak{h}$, i.e. the complement of the reflection hyperplanes. To describe $\mathcal{D}(\mathfrak{g})//\mathfrak{g}$, we will construct a homomorphism $\mathrm{HC} : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{D}(\mathfrak{h})^W$, called the Harish-Chandra homomorphism (as it was first constructed by Harish-Chandra). Recall that we have the classical Harish-Chandra isomorphism $\zeta : \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W$, defined simply by restricting $\mathfrak{g}$-invariant functions on $\mathfrak{g}$ to the Cartan subalgebra $\mathfrak{h}$. Using this isomorphism, we can define an action of $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ on $\mathbb{C}[\mathfrak{h}]^W$, which is clearly given by W-invariant differential operators. However, these operators will, in general, have poles on the reflection hyperplanes. Thus we get a homomorphism $\mathrm{HC}' : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^W$.

The homomorphism HC' is called the radial part homomorphism, as for example for $\mathfrak{g} = \mathfrak{su}(2)$ it computes the radial parts of rotationally invariant differential operators on $\mathbb{R}^3$ in spherical coordinates. This homomorphism is not yet what we want, since it does not actually land in $\mathcal{D}(\mathfrak{h})^W$ (the radial parts have poles).

Thus we define the Harish-Chandra homomorphism by twisting HC' by the discriminant $\delta(\mathbf{x}) = \prod_{\alpha>0}(\alpha, \mathbf{x})$ ($\mathbf{x} \in \mathfrak{h}$, and α runs over positive roots of $\mathfrak{g}$):

$$\mathrm{HC}(D) := \delta \circ \mathrm{HC}'(D) \circ \delta^{-1} \in \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^W.$$

Theorem 10.1. (i) (*Harish-Chandra,* [HC]) *For any reductive* $\mathfrak{g}$*,* HC *lands in* $\mathcal{D}(\mathfrak{h})^W \subset \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^W$.

(ii) (*Levasseur-Stafford* [LS]) *The homomorphism* HC *defines an isomorphism* $\mathcal{D}(\mathfrak{g})//\mathfrak{g} = \mathcal{D}(\mathfrak{h})^W$.

Remark 10.2. (1) Part (i) of the theorem says that the poles magically disappear after conjugation by δ.

(2) Both parts of this theorem are quite nontrivial. The first part was proved by Harish-Chandra using analytic methods, and the second part by Levasseur and Stafford using the theory of $\mathcal{D}$-modules.

In the case $\mathfrak{g} = \mathfrak{gl}_n$, Theorem 10.1 is a quantum analog of Theorem 9.14. The remaining part of this subsection is devoted to the proof of Theorem 10.1 in this special case, using Theorem 9.14.

We start the proof with the following proposition, valid for any reductive Lie algebra.

Proposition 10.3. *If $D \in (S\mathfrak{g})^{\mathfrak{g}}$ is a differential operator with constant coefficients, then* $\mathrm{HC}(D)$ *is the W-invariant differential operator with constant coefficients on $\mathfrak{h}$, obtained from D via the classical Harish-Chandra isomorphism $\eta : (S\mathfrak{g})^{\mathfrak{g}} \to (S\mathfrak{h})^W$.*

Proof. Without loss of generality, we may assume that $\mathfrak{g}$ is simple.

Lemma 10.4. *Let D be the Laplacian $\Delta_{\mathfrak{g}}$ of $\mathfrak{g}$, corresponding to an invariant form. Then* $\mathrm{HC}(D)$ *is the Laplacian $\Delta_{\mathfrak{h}}$.*

Proof. Let us calculate $\mathrm{HC}'(D)$. We have

$$D = \sum_{i=1}^{r} \partial_{x_i}^2 + 2\sum_{\alpha>0} \partial_{f_\alpha}\partial_{e_\alpha},$$

where x_i is an orthonormal basis of $\mathfrak{h}$, and e_α, f_α are root elements such that $(e_\alpha, f_\alpha) = 1$. Thus if $F(\mathbf{x})$ is a $\mathfrak{g}$-invariant function on $\mathfrak{g}$, then we get

$$(DF)|_{\mathfrak{h}} = \sum_{i=1}^{r} \partial_{x_i}^2 (F|_{\mathfrak{h}}) + 2\sum_{\alpha>0} (\partial_{f_\alpha}\partial_{e_\alpha} F)|_{\mathfrak{h}}.$$

Now let $\mathbf{x} \in \mathfrak{h}$, and consider $(\partial_{f_\alpha}\partial_{e_\alpha} F)(\mathbf{x})$. We have

$$(\partial_{f_\alpha}\partial_{e_\alpha} F)(\mathbf{x}) = \partial_s\partial_t|_{s=t=0} F(\mathbf{x} + tf_\alpha + se_\alpha).$$

On the other hand, we have

$$\mathrm{Ad}(\mathbf{e}^{s\alpha(\mathbf{x})^{-1}e_\alpha})(\mathbf{x} + tf_\alpha + se_\alpha) = \mathbf{x} + tf_\alpha + ts\alpha(\mathbf{x})^{-1}h_\alpha + \cdots,$$

where $h_\alpha = [e_\alpha, f_\alpha]$. Hence, $\partial_s\partial_t|_{s=t=0} F(\mathbf{x} + tf_\alpha + se_\alpha) = \alpha(\mathbf{x})^{-1}(\partial_{h_\alpha} F)(\mathbf{x})$. This implies that

$$\mathrm{HC}'(D)F(\mathbf{x}) = \Delta_{\mathfrak{h}} F(\mathbf{x}) + 2\sum_{\alpha>0} \alpha(\mathbf{x})^{-1}\partial_{h_\alpha} F(\mathbf{x}).$$

Now the statement of the Lemma reduces to the identity $\delta^{-1} \circ \Delta_{\mathfrak{h}} \circ \delta = \Delta_{\mathfrak{h}} + 2\sum_{\alpha>0} \alpha(\mathbf{x})^{-1}\partial_{h_\alpha}$. This identity follows immediately from the identity $\Delta_{\mathfrak{h}}\delta = 0$. To prove the latter, it suffices to note that δ is the lowest degree nonzero polynomial on $\mathfrak{h}$, which is antisymmetric under the action of W. The lemma is proved. □

Now let D be any element of $(S\mathfrak{g})^{\mathfrak{g}} \subset \mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ of degree d (operator with constant coefficients). It is obvious that the leading order part of the operator $\mathrm{HC}(D)$ is the operator $\eta(D)$ with constant coefficients, whose symbol is just the restriction of the symbol of D from $\mathfrak{g}^*$ to $\mathfrak{h}^*$. Our job is to show that in fact $\mathrm{HC}(D) = \eta(D)$. To do so, denote by Y the difference $\mathrm{HC}(D) - \eta(D)$. Assume $Y \neq 0$. By Lemma 10.4, the operator $\mathrm{HC}(D)$ commutes with $\Delta_{\mathfrak{h}}$. Therefore, so does Y. Also Y has homogeneity degree d but order $m \leq d-1$. Let $S(\mathbf{x}, \mathbf{p})$ be the symbol of Y ($\mathbf{x} \in \mathfrak{h}, \mathbf{p} \in \mathfrak{h}^*$). Then S is a homogeneous function of homogeneity degree d under the transformations $\mathbf{x} \to t^{-1}\mathbf{x}$, $\mathbf{p} \to t\mathbf{p}$, polynomial in $\mathbf{p}$ of degree m. From these properties of S it is clear that S is not a polynomial (its degree in $\mathbf{x}$ is $m - d < 0$). On the other hand, since Y commutes with $\Delta_{\mathfrak{h}}$, the Poisson bracket of S with $\mathbf{p}^2$ is zero. Thus Proposition 10.3 follows from Lemma 2.22. □

Now we continue the proof of Theorem 10.1. Consider the filtration on $\mathcal{D}(\mathfrak{g})$ in which $\deg(\mathfrak{g}) = 1 \deg(\mathfrak{g}^*) = 0$ (the order filtration), and the associated graded map $\mathrm{grHC} : \mathbb{C}[\mathfrak{g} \times \mathfrak{g}^*]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*]^W$, which attaches to every differential operator the symbol of its radial part. It is easy to see that this map is just the restriction map to $\mathfrak{h} \oplus \mathfrak{h}^* \subset \mathfrak{g} \oplus \mathfrak{g}^*$, so it actually lands in $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$.

Moreover, grHC is a map **onto** $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$. Indeed, grHC is a Poisson map, so the surjectivity follows from the following Lemma.

Lemma 10.5. *$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ is generated as a Poisson algebra by $\mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[\mathfrak{h}^*]^W$, i.e. by functions $f_m = \sum x_i^m$ and $f_m^* = \sum p_i^m$, $m \geq 1$.*

Proof. We have $\{f_m^*, f_r\} = mr \sum x_i^{r-1} p_i^{m-1}$. Thus the result follows from Corollary 9.13. $\square$

Let K_0 be the kernel of grHC. Then by Theorem 9.14, K_0 is the ideal of the commuting scheme $\mathrm{Comm}(\mathfrak{g})/G$.

Now consider the kernel K of the homomorphism HC. It is easy to see that $K \supset J^{\mathfrak{g}}$, so $\mathrm{gr}(K) \supset \mathrm{gr}(J)^{\mathfrak{g}}$. On the other hand, since K_0 is the ideal of the commuting scheme, we clearly have $\mathrm{gr}(J)^{\mathfrak{g}} \supset K_0$, and $K_0 \supset \mathrm{gr}K$. This implies that $K_0 = \mathrm{gr}K = \mathrm{gr}(J)^{\mathfrak{g}}$, and $K = J^{\mathfrak{g}}$.

It remains to show that $\mathrm{Im}\,\mathrm{HC} = \mathcal{D}(\mathfrak{h})^W$. Since $\mathrm{gr}K = K_0$, we have $\mathrm{grIm}\,\mathrm{HC} = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$. Therefore, to finish the proof of the Harish-Chandra and Levasseur-Stafford theorems, it suffices to prove the following proposition.

Proposition 10.6. *$\mathrm{Im}\,\mathrm{HC} \supset \mathcal{D}(\mathfrak{h})^W$.*

Proof. We will use the following Lemma.

Lemma 10.7 (N. Wallach, [Wa]). *$\mathcal{D}(\mathfrak{h})^W$ is generated as an algebra by W-invariant functions and W-invariant differential operators with constant coefficients.*

Proof. The lemma follows by taking associated graded algebras from Lemma 10.5. $\square$

Remark 10.8. Levasseur and Stafford showed [LS] that this lemma is valid for any finite group W acting on a finite dimensional vector space $\mathfrak{h}$. However, the above proof does not apply, since, as explained in [Wa], Lemma 10.5 fails for many groups W, including Weyl groups of exceptional Lie algebras E_6, E_7, E_8 (in fact it even fails for the cyclic group of order > 2 acting on a 1-dimensional space!). The general proof is more complicated and uses some results in noncommutative algebra.

Lemma 10.7 and Proposition 10.3 imply Proposition 10.6. $\square$

Thus, Theorem 10.1 is proved.

10.3. **Corollaries of Theorem 10.1.** Let $\mathfrak{g}_{\mathbb{R}}$ be the compact form of $\mathfrak{g}$, and $\mathcal{O}$ a regular coadjoint orbit in $\mathfrak{g}_{\mathbb{R}}^*$. Consider the map

$$\psi_{\mathcal{O}} : \mathfrak{h} \to \mathbb{C}, \quad \psi_{\mathcal{O}}(\mathbf{x}) = \int_{\mathcal{O}} \mathbf{e}^{(\mathbf{b},\mathbf{x})} \mathrm{d}\mathbf{b}, \ \mathbf{x} \in \mathfrak{h},$$

where $\mathrm{d}\mathbf{b}$ is the measure on the orbit coming from the Kirillov-Kostant symplectic structure.

Theorem 10.9 (Harish-Chandra formula)**.** *For a regular element* $\mathbf{x} \in \mathfrak{h}$*, we have*

$$\psi_{\mathcal{O}}(\mathbf{x}) = \delta^{-1}(\mathbf{x}) \sum_{w \in W} (-1)^{\ell(w)} \mathbf{e}^{(w\lambda, \mathbf{x})},$$

where λ *is the intersection of* $\mathcal{O}$ *with the dominant chamber in the dual Cartan subalgebra* $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{g}^*_{\mathbb{R}}$*, and* $\ell(w)$ *is the length of an element* $w \in W$*.*

Proof. Take $D \in (S\mathfrak{g})^{\mathfrak{g}}$. Then $\delta(\mathbf{x})\psi_{\mathcal{O}}$ is an eigenfunction of $\mathrm{HC}(D) = \eta(D) \in (S\mathfrak{h})^W$ with eigenvalue $\chi_{\mathcal{O}}(D)$, where $\chi_{\mathcal{O}}(D)$ is the value of the invariant polynomial D at the orbit $\mathcal{O}$.

Since the solutions of the equation $\eta(D)\varphi = \chi_{\mathcal{O}}(D)\varphi$ have a basis $\mathbf{e}^{(w\lambda,\mathbf{x})}$ where $w \in W$, we have

$$\delta(\mathbf{x})\psi_{\mathcal{O}}(\mathbf{x}) = \sum_{w \in W} c_w \cdot \mathbf{e}^{(w\lambda, \mathbf{x})}.$$

Since it is antisymmetric, we have $c_w = c \cdot (-1)^{\ell(w)}$, where c is a constant. The fact that $c = 1$ can be shown by comparing the asymptotics of both sides as $\mathbf{x} \to \infty$ in the regular chamber (using the stationary phase approximation for the integral). □

From Theorem 10.9 and the Weyl Character formula, we have the following corollary.

Corollary 10.10 (Kirillov character formula for finite dimensional representations, [Ki])**.** *If* λ *is a dominant integral weight, and* L_λ *is the corresponding representation of* G*, then*

$$\mathrm{Tr}_{L_\lambda}(\mathbf{e}^{\mathbf{x}}) = \frac{\delta(\mathbf{x})}{\delta_{\mathrm{Tr}}(\mathbf{x})} \int_{\mathcal{O}_{\lambda+\rho}} \mathbf{e}^{(\mathbf{b},\mathbf{x})} d\mathbf{b},$$

where $\delta_{\mathrm{Tr}}(\mathbf{x})$ *is the trigonometric version of* $\delta(\mathbf{x})$*, i.e. the Weyl denominator* $\prod_{\alpha>0}(\mathbf{e}^{\alpha(\mathbf{x})/2} - \mathbf{e}^{-\alpha(\mathbf{x})/2})$*, and* $\mathcal{O}_\mu$ *denotes the coadjoint orbit passing through* μ*.*

10.4. **The deformed Harish-Chandra homomorphism.** Finally, we would like to explain how to quantize the Calogero-Moser space $\mathcal{C}_n$, using the procedure of quantum Hamiltonian reduction.

Let $\mathfrak{g} = \mathfrak{gl}_n$, $A = \mathcal{D}(\mathfrak{g})$ as above. Let k be a complex number, and W_k be the representation of $\mathfrak{sl}_n$ on the space of functions of the form $(x_1 \cdots x_n)^k f(x_1, \ldots, x_n)$, where f is a Laurent polynomial of degree 0. We regard W_k as a $\mathfrak{g}$-module by pulling it back to $\mathfrak{g}$ under the natural projection $\mathfrak{g} \to \mathfrak{sl}_n$. Let I_k be the annihilator of W_k in $U(\mathfrak{g})$. The ideal I_k is the quantum counterpart of the coadjoint orbit of matrices T such that $T + 1$ has rank 1.

Let $B_k = \mathcal{D}(\mathfrak{g})^{\mathfrak{g}}/(\mathcal{D}(\mathfrak{g})\mu(I_k))^{\mathfrak{g}}$ where $\mu : U(\mathfrak{g}) \to A$ is the quantum momentum map (the quantum Hamiltonian reduction with respect to the ideal I_k). Then B_k has a filtration induced from the order filtration of $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$.

Let $\mathrm{HC}_k : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \to B_k$ be the natural homomorphism, and $K(k)$ be the kernel of HC_k.

Theorem 10.11 (Etingof-Ginzburg, [EG])**.** (i) $K(0) = K$*,* $B_0 = \mathcal{D}(\mathfrak{h})^W$*,* $\mathrm{HC}_0 = \mathrm{HC}$*.*

(ii) $\mathrm{gr}K(k) = \mathrm{Ker}\,(\mathrm{gr}\mathrm{HC}_k) = K_0$ *for all complex* k*. Thus,* HC_k *is a flat family of homomorphisms.*

(iii) *The algebra* $\mathrm{gr}B_k$ *is commutative and isomorphic to* $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ *as a Poisson algebra.*

Because of this theorem, the homomorphism HC_k is called the deformed Harish-Chandra homomorphism.

Theorem 10.11 implies that B_k is a quantization of the Calogero-Moser space $\mathcal{C}_n$ (with deformation parameter $1/k$). But we already know one such quantization - the spherical

Cherednik algebra $B_{1,k}$ for the symmetric group. Therefore, the following theorem comes as no surprize.

Theorem 10.12 ([EG]). *The algebra B_k is isomorphic to the spherical rational Cherednik algebra $B_{1,k}(\mathfrak{S}_n, \mathbb{C}^n)$.*

Thus, quantum Hamiltonian reduction provides a Lie-theoretic construction of the spherical rational Cherednik algebra for the symmetric group. A similar (but more complicated) Lie theoretic construction exists for symplectic reflection algebras for wreath product groups defined in Example 8.5 (see [EGGO]).

10.5. **Notes.** Our exposition in this section follows Section 4, Section 5 of [E4].

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