18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 7

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1. Ruled Surfaces (contd.)

As before, we have a short exact sequence $0 \to L \to E \to M \to 0$ for any vector bundle of rank 2 on a curve B. Let $\deg(E) = \deg(\wedge^2 E) = \deg L \deg M$, $h^i(E) = \dim H^i(B, E)$. We can twist E by some line bundle so that $h^0(B, E) \neq 0$ but for any line bundle M on C with $\deg(M) < 0$, $h^0(E \otimes M) = 0$. Such and E is called normalized. The number $e = -\deg(E)$ is called the invariant of the ruled surface $X = \mathbb{P}(E)$. There is a section $\sigma : B \to X$ with image B_0 s.t. $\mathcal{O}_\lambda(B_0) = \mathcal{O}_X(1)$.

Proof. Let $s \in H^0(E)$ be a nonzero section. This gives a map $0 \to \mathcal{O}_B \xrightarrow{s} E \to E/s\mathcal{O}_B = L \to 0$. We claim that L is an invertible sheaf on B. If not, then L must have torsion. Let $F \subset E$ be the inverse image of the torsion subsheaf of L. F is torsion free of rank 1 on C. By assumption, $\mathcal{O}_B \subsetneq F$, so $\deg F > 0$. But then $H^0(E \otimes F^{-1}) \neq 0$ and $\deg(F^{-1}) < 0$ contradicting that E is normalized. Thus, L must have been invertible. The universal property then gives us a section $\sigma_0 : B \to X$ with image B_0. Then it is easy to check that $\mathcal{O}_X(B_0) \cong \mathcal{O}_X(1)$. □

Lemma 1. *Let X be a ruled surface over a curve B of genus g, determined by a normalized E of rank 2. Then*

(a) *if E decomposes, then $E \cong \mathcal{O}_B \oplus L$ for some L with $\deg L \leq 0$ so $e = -\deg E = -\deg L \geq 0$, and*
(b) *if E is indecomposable, then $-2g \leq e \leq 2g - 2$.*

Corollary 1. *Every E of rank 2 on $\mathbb{P}^1$ decomposes (i.e. no case (b)).*

Thus, a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ can be written as $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. In fact, by a theorem of Grothendieck, every locally free sheaf on $\mathbb{P}^1$ is decomposable.

Proof. (a) If E decomposes, then $E = L \oplus M$, where L and M are line bundles on B. Then we must have $\deg L, \deg M \leq 0$. (Otherwise, $E \otimes L^{-1}$ or $E \otimes M^{-1}$ would have global sections, contradicting the fact that E is normalized.) Also, $H^0(E) = H^0(M) \oplus H^0(L)$, so that L or M has global sections and therefore must be $\mathcal{O}_B$, since its degree is positive.

(b) We have $0 \to \mathcal{O}_B \to E \to L \to 0$. This is a nontrivial extension, so it corresponds to a nontrivial element of $\mathrm{Ext}^1(L, \mathcal{O}_B) = H^1(B, L^{-1})$. So

$$\begin{aligned} H^1(B, L^{-1}) = H^0(B, L + K_B) \neq 0 &\implies \deg(L + K_B) \geq 0 \\ &\implies e = -\deg E = -\deg L = \deg L^{-1} \leq 2g - 2 \end{aligned} \tag{1}$$

On the other hand, $H^0(E \otimes M) = 0$ for all M with $\deg M < 0$. Take an M with $\deg M = -1$. Then we get

$$0 = H^0(E \otimes M) \to H^0(L \otimes M) \to H^1(M) \to \cdots \tag{2}$$

implying that $h^0(L \otimes M) \leq h^1(M)$. Now, since $\deg M < 0, h^0(M) = 0$. By Riemann-Roch, $h^1(M) = g$ and $h^0(L \otimes M) \geq \deg L - 1 + 1 - g = \deg L - g \implies \deg L \leq 2g \implies e \geq -2g$ as desired. □

1.1. **Invariants.** Let $\mathbb{P}_B(E) = X \to B$ be a ruled surface.

Proposition 1. *For h the class of $\mathcal{O}_X(1)$,*

1. $h^2 = \deg(E) = -e$.
2. $K \sim -2h + \pi^*(K_B + [\wedge^2 E]), K \equiv -2h + (2g - 2 + \deg(E))f$.
3. $K^2 = 8(1 - g)$.
4. $q = h^1(X, \mathcal{O}_X) = g, p_g = h^2(X, \mathcal{O}_X) = 0$ *and* $p_n = h^0(X, \omega_X^{\otimes n}) = 0$ *for all* $n > 0$.

Proof. (1) Let E' be a vector bundle on a surface s.t. there exists line bundles L, M satisfying $0 \to L \to E' \to M \to 0$. Then

$$\begin{aligned} L \cdot M = L^{-1} \cdot M^{-1} &= \chi(\mathcal{O}_X) - \chi(L) - \chi(M) + \chi(L \otimes M) \\ &= \chi(\mathcal{O}_X) - \chi(E') + \chi(\wedge^2 E') \end{aligned} \tag{3}$$

so $L \cdot M$ only depends on E', call it $c_2(E')$. This is actually the second Chern class: the total Chern classes of L and M are $(1 + L)$ and $(1 + M)$ respectively. Apply this to $\pi^* E$ with $0 \to \pi^* L \to \pi^* E \to \pi^* M \to 0$. We get $c_2(\pi^* E) = \pi^* L \cdot \pi^* M = 0$ (multiple of the fiber). We also have $0 \to N \to \pi^* E \to \mathcal{O}_X(1) \to 0$ (corresponding to the section), implying that $\mathcal{O}_X(1) \cdot N = 0 \implies h \cdot N = 0$. Moreover, $\pi^* \wedge^2 E \cong N \otimes \mathcal{O}_X(1)$. $N \sim -h + \pi^*[\wedge^2 E]$. Thus,

$$0 = hN = -h^2 + h\pi^*[\wedge^2 E] = \deg E - h^2 \tag{4}$$

and $h^2 = \deg E$.

(2) Let $K_X \sim ah + \pi^* b$ for $b \in \mathrm{Pic}\, B$. By adjunction for a fiber f, we have $-2 = 2g - 2 = f \cdot (f + K) = f \cdot (ah + \pi^* b) = a$, so $\omega_X \sim \mathcal{O}_X(-2B_0 + \pi^* b)$. Next, using the adjunction formula for B_0 (section corresponding to $\mathcal{O}_X(1)$) gives

$$\omega_B = \omega_X \otimes \mathcal{O}_X(B_0) \otimes \mathcal{O}_{B_0} \cong \mathcal{O}_X(-B_0 + \pi^* b) \otimes \mathcal{O}_{B_0} \tag{5}$$

Identifying B_0 with B using π, we get $K_B = -[\wedge^2 E] + b \implies b = K_B + [\wedge^2 E]$ via

$$0 \to \mathcal{O}_B \to E \to E/\mathcal{O}_B \to 0 \tag{6}$$

and $\wedge^2 E = E/\mathcal{O}_B = \pi^*(\mathcal{O}_X(1)) = \mathcal{O}_X(B_0)|_{B_0}$. Thus, $K \sim -2h + \pi^*(K_B + [\bigwedge^2 E])$ and $K \equiv -2h + (2g - 2 + \deg E)f$ as desired.

(3) $K^2 = 4h^2 - 4(h \cdot f)(2g-2+\deg E) = 8(1-g)$ since $h \cdot f = 1$ and $h^2 = \deg E$.

(4) q, p_g, p_n are birational invariants, so we can assume $X = B \times \mathbb{P}^1$. Then $H^0(X, \Omega^1_X) = H^0(B, \omega_B) \oplus H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1})$ and $q = g$ (since the latter term vanishes). This shows that $h^{10} = g \implies h^{01} = g$ as long as we work over $\mathbb{C}$, implying that $H^0(X, \Omega_X) = 1$. More generally, we use Noether's formula: $\chi(\mathcal{O}_X) = \frac{1}{12}(K^2 + c_2)$, where $c_2(X) = c_2(B) \cdot c_2(\mathbb{P}^2) = (2-2g)2 = 4 - 4g$, implying that $\chi(\mathcal{O}_X = 1 - g$ and $h^1(\mathcal{O}_X) = g$ since $h^2(\mathcal{O}_X) = h^0(\omega_X) = 0$. Moreover,

$$H^0(X, \omega_X^{\otimes n}) = H^0(B, \omega_B^{\otimes n}) \otimes H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes n}) = 0 \tag{7}$$

as stated. □

Remark. For more results on vector bundles of rank 2, see Hartshorne, Beauville, etc. Here are some results: for B an elliptic curve, vectors bundles of rank 2 are either

- decomposable
- isomorphic to $E \otimes L$ for E a nontrivial extension of $\mathcal{O}_B$ by $\mathcal{O}_B$, i.e. a nonzero element of $H^1(B\mathcal{O}_B) \cong k$
- isomorphic to $E \otimes L$ for E a nontrivial extension of $\mathcal{O}_B(p)$ by $\mathcal{O}_B$, i.e. a nonzero element of $\operatorname{Ext}^1(\mathcal{O}_B(p), \mathcal{O}_B) = H^1(B, \mathcal{O}_B(-p)) \cong k$ by Riemann-Roch.

For $g \geq 2$, there are $3g - 3$ moduli. More precisely, one looks at semi-stable vector bundles of rank 2: $\mathcal{E}$ is semi-stable if for every quotient locally-free sheaf $\mathcal{E} \to \mathcal{F} \to 0$, we have $\frac{\deg \mathcal{F}}{\operatorname{rk} \mathcal{F}} \geq \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}}$.

1.2. **Elementary Transformations.**

Let $\pi : X \to B$ be a given geometrically ruled surface. Let $p \in X$ and let E be the fiber of π containing p. Let $f : \tilde{X} \to X$ be the blowup of X at p. Then since $F^2 = 0$, the proper transform $\tilde{F}$ satisfies $\tilde{F}^2 = -1$. So it is an exceptional curve of the first kind, so we can blow it down to get $\pi' : X' \to B$ another geometrically ruled surface.

Now, let $X = \mathbb{P}_B(E)$ for E a rank 2 vector bundle on B. The point $p \in X$ corresponds to a surjection $u_p : E \to k(p)$ to the skyscraper sheaf at p (by the universal property). $\operatorname{Ker} u_p = E'$, a vector bundle of rank 2, so set $X' = \mathbb{P}_B(E')$. Such an $X \dashrightarrow X'$ is an *elementary transformation*, and corresponds to $E' \to E$.

Problem. Let X be a minimal ruled surface over a curve B of genus > 0. Then X is obtained from $B \times \mathbb{P}^1$ by a finite sequence of elementary transformations.

2. Rational Surfaces

Rational surfaces are surfaces birational to $\mathbb{P}^2$ or to $\mathbb{P}^1 \times \mathbb{P}^1$. We've already shown that any minimal model must be geometrically ruled over $\mathbb{P}^1$. So let's check which $\mathbb{P}_{\mathbb{P}^1}(E)$ are minimal. We showed that E can be twisted so that it becomes $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$, $n \geq 0$. Its projectivization is called the n-th *Hirzebruch surface* $\mathbb{F}_n$. As usual, let h be the class of $\mathcal{O}_{\mathbb{F}_n}(1)$ in Pic $\mathbb{F}_n$, f the class of a fiber.

Proposition 2. (a) Pic $\mathbb{F}_n = \mathbb{Z}h \oplus \mathbb{Z}f, f^2 = 0, fh = 1, h^2 = n$,

(b) *if $n > 0$, there is a unique irreducible curve B on $\mathbb{F}_n$ with negative self intersection, and b is its closure in* Pic $\mathbb{F}_n$*, then $b = h - nf, b^2 = -n$, and*

(c) *$\mathbb{F}_n$ and $\mathbb{F}_m$ are not isomorphic unless $m = n$, $\mathbb{F}_n$ is minimal except when $n = 1$, and $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blown up at a point.*

Proof. (a) follows from the previous results, noting that $h^2 = \deg(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. For (b), let s be the section of $\pi : \mathbb{F}_n \to \mathbb{P}^1$ corresponding to $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{O}_{\mathbb{P}^1}$. Let $B = s(\mathbb{P}^1)$ and b its class. Then $b = \ell h + mf$ for $\ell, m \in \mathbb{Z}$. Since $b \cdot f = 1$, $\ell = 1, b = h + nf$.

$$s^* \mathcal{O}_{\mathbb{F}_n}(1) = \mathcal{O}_{\mathbb{P}^1} \implies h \cdot b = \deg_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}) = 0 \tag{8}$$

Thus,

$$0 = h(h + mf) = h^2 + m \implies m = -h^2 = -n \implies b = h - nf \tag{9}$$

and $b^2 = (h - nf)^2 = h^2 - 2n(h \cdot f) + n^2 f^2 = n - 2n + 0 = -n$. We need to show it's the only irreducible curve with negative self-intersection. Let C be some irreducible curve $\neq B$. Write $[C] = \alpha h + \beta f$. Then $C \cdot f \geq 0$ by the useful lemma, since $f^2 = 0$ is nonnegative, implying that $\alpha \geq 0$ and $C \cdot b \geq 0 \implies \beta \geq 0$ since $b \cdot f = 1$ and $b \cdot h = 0$. Thus, $[C]^2 = \alpha^2 n + 2\alpha\beta \geq 0$. The induced form on Pic $\mathbb{F}_n = \mathfrak{X}f + \mathbb{Z}b$ can be written as $\begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}$.

For (c), it follows from the existence of the special curve of self-intersection $-n$ on $\mathbb{F}_n$ that $\mathbb{F}_n \neq \mathbb{F}_m$ if $n \neq m$. Note that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and any $C = \alpha h_1 + \beta h_2$ so all curves on $\mathbb{F}_0$ have non-negative self-intersection. There are no -1 curves on $\mathbb{F}_n$ for $n \neq 1$, so $\mathbb{F}_n$ is minimal for $n \neq 1$. For $\mathbb{F}_1$, let S be the blowup of $\mathbb{P}^2$ at 0, E the exceptional divisor. Projection away from 0 defines a morphism $\pi : S \to \mathbb{P}^1$ which gives S as a geometrically ruled surface on $\mathbb{P}^1$. $E^2 = -1$ implies $S \cong \mathbb{F}_1$, so $\mathbb{F}_1$ is not minimal, and is isomorphic to $\mathbb{P}^2$ blown up at a point. $\square$

Note. An elementary transformation of $\mathbb{F}_n$ corresponding to a point on a special curve gives $\mathbb{F}_{n+1}$, while one corresponding to a point not on the special curve gives $\mathbb{F}_{n-1}$.