18.727 Topics in Algebraic Geometry: Algebraic Surfaces
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ALGEBRAIC SURFACES, LECTURE 1

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1. INTRODUCTION

This course concerns algebraic surfaces, which for our purposes will be projective and non-singular over a field k. Usually, we will assume k is algebraically closed. The simplest example of algebraic surfaces arise as hypersurfaces in $\mathbb{P}^3$. Let $S = V(f)$ for f an irreducible, homogeneous polynomial of degree d: abstractly, $S = \operatorname{Proj} k[X, Y, Z, W]/(f)$.

$d = 1$ We can change coordinates so that $f = X$, giving an isomorphism to the rational surface $\mathbb{P}^2$.

$d = 2$ Changing coordinates, we can write $f = XY - ZW$, and obtain an isomorphism to the rational ruled surface $\mathbb{P}^1 \times \mathbb{P}^1$: the two rulings are given by $X = \lambda W, \lambda Y = Z$ or $X = \lambda Z, \lambda Y = W$. Note that the particular surface may have many fewer rational points than $\mathbb{P}^1 \times \mathbb{P}^1$, e.g. $X^2 + Y^2 + Z^2 + W^2 = 0$ has no rational solutions. The smooth quadric in $\mathbb{P}^3$ is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$; it is isomorphic to $\mathbb{P}^2(2-1)$, i.e. $\mathbb{P}^2$ blown up in 2 points, with the proper transform of the line joining them blown down.

$d = 3$ Cubic surfaces in $\mathbb{P}^3$ are also rational surfaces, realized as $\mathbb{P}^2$ blown up at 6 points. Each has 27 lines, which can be seen explicitly in the case of $X^3 + Y^3 + Z^3 + W^3 = 0$: nine of the lines are given by $X + \omega Y = 0, Z + \sigma W = 0$, where ω, σ are cubic roots of unity. Similarly we have the lines given by $X + \omega Z = 0, Y + \sigma W = 0$ and $X + \omega W = 0, Y + \sigma Z = 0$. The configuration of 27 lines and their intersection points is called the Schäfli graph. It is strongly regular, with parameters $(27, 16, 10, 8)$. To see that it is rational, choose two non-intersecting lines L, M: taking projections gives $\phi_L : X \to \mathbb{P}^1, \phi_M : X \to \mathbb{P}^1 \implies \phi_L \times \phi_M : X \to \mathbb{P}^1 \times \mathbb{P}^1$ which is birational by Bezout.

$d = 4$ Quartic surfaces in $\mathbb{P}^3$ are examples of $K3$ surfaces (over $\mathbb{C}$, these are examples of Calabi-Yau manifolds). This class includes, for instance, Kummer surfaces of abelian surfaces, which are classically well-studied.

For $K3$ surfaces, the geometry and moduli are well-known, but the arithmetic less so. For instance, the number of parameters for a quartic surface in $\mathbb{P}^3$ is $\binom{4+3}{3} - \dim \mathrm{GL}_4 = 35 - 16 = 19$.

$d \geq 5$ These are surfaces of general type.

2. Rough Plan for the Course

The goal will be a full classification of surfaces, mostly with proofs.

(1) Preliminaries: intersection theory, Riemann-Roch, Hodge index theorem
(2) Birational maps, minimal models
(3) Classification
- In characteristic 0, we have the Enriques-Castelnuovo-Zariski-Kodaira classification of minimal models of surfaces based on Kodaira dimension, $\kappa(S) =$ the maximal dimension of the image of S under the linear system nK_S ($-\infty$ if this linear system is always 0), geometric genus $p_g = h^0(S, K_S)$, and irregularity $q = h^1(S, \mathcal{O}_S)$. Here, K_S is divisor class corresponding to the canonical sheaf $\omega = \wedge^2 \Omega^1$.

$\kappa(S) = -\infty$: These are ruled (e.g. rational) surfaces, i.e. there is a map $S \to C$ onto a curve with generic fiber smooth of genus 0, and have geometric genus 0 (in fact, all their plurigenera are zero).

$\kappa(S) = 0$: There are four possibilities based on combinations of (p_g, q):
$(0, 0)$ are Enriques surfaces,
$(0, 1)$ are bielliptic or hyperelliptic surfaces,
$(1, 0)$ are K3 surfaces, and
$(1, 2)$ are abelian surfaces.

$\kappa(S) = 1$: These are "honest" elliptic surfaces. An elliptic surfaces is one with an elliptic fibration, i.e. a map $S \to C$ onto a curve with generic fiber smooth of genus 1.

$\kappa(S) = 2$: These are surfaces of general type.

- In characteristic > 0, we have the Bombieri-Mumford classification of minimal models, which require the ℓ-adic Betti numbers $b_i = \dim_{\mathbb{Q}_\ell} H^i(X, \mathbb{Q}_\ell)$ for $\ell \neq \mathrm{char}(k)$. Note that b_i is independent of ℓ and agrees with the dimension of $H^i(X, \mathbb{C})$ for a smooth complex variety X.

$\kappa(S) = -\infty$: These are ruled surfaces. By Castelnuovo's theorem, only rational if $q = p_2 = 0$ (works in any characteristic).

$\kappa(S) = 0$: There are four possibilities based on combinations of (b_1, b_2):
$(0, 10)$ are Enriques surfaces, "classical" if $(p_g, q) = (0, 0)$ and "non-classical" if $(p_g, q) = (1, 1)$.
$(2, 2)$ are bielliptic or hyperelliptic surfaces if $(p_g, q) = (0, 1)$, and quasi-hyperelliptic surfaces if $(p_g, q) = (1, 2)$.
$(0, 22)$ are $K3$ surfaces, and

$(4, 6)$ are abelian surfaces.

$\kappa(S) = 1$: These are surfaces with elliptic or quasi-elliptic fibrations: in the latter case, the generic fiber has arithmetic genus 1 but has a cusp (only exists in characteristics $2, 3$, e.g. $y^2 = x^3 + t$).

$\kappa(S) = 2$: These are surfaces of general type.

(4) We will discuss various aspects of the geometry and arithmetic of surfaces as they arise, and some singularity theory and other topics according to interest.

2.1. References.

- Beauville, *Complex Algebraic Surfaces.*
- Badescu, *Algebraic Surfaces,*
- Barth et al, *Compact Complex Surfaces*
- Reid, *Chapters on Algebraic Surfaces* in PCMI vol. 3
- Hartshone, *Algebraic Geometry*, chapter 5
- Griffiths and Harris, *Principles of Algebraic Geometry*, chapter 4
- Kodaira, *On Compact Complex Analytic Surfaces I,II,III*, in the Annals of Mathematics
- Bombieri and Mumford, *Enriques' Classification of Surfaces in Characteristic p*, Invent. Math

3. Preliminaries

Let $X = S$ be a nonsingular, projective algebraic surface over an algebraically closed field k. We recall the basic notions of intersection theory on surfaces.

Definition 1. *A* curve *on S is a closed, integral subscheme of (co)dimension 1. A* divisor *is a formal sum of curves with multiplicity, and is* effective *if the coefficients are nonnegative. The set of divisors form a group* $\operatorname{Div} X$. *For C, D distinct curves on X, the intersection multiplicity of C and D at $p \in C \cap D$ is $m_p(C \cap D) = \dim_k \mathcal{O}_p/(f, g)$ where f is an equation for C in $\mathcal{O}_p$ and g is an equation for D in $\mathcal{O}_p$. C and D are called* transverse *if $m_p(C, D) = 1$, i.e. f, g span the maximal ideal. The* intersection product *between C, D is $C \cdot D = \sum_{p \in C \cap D} m_p(C \cap D)$, and extends to divisors in the obvious manner.*

Recall that the ideal sheaf defining C is $\mathcal{O}_X(-C)$. Let $\mathcal{O}_{C \cap D} = \mathcal{O}_X/(\mathcal{O}_X(-C) + \mathcal{O}_X(-D))$, a skyscraper sheaf concentrated on $C \cap D$. At each $p \in C \cap D$, $(\mathcal{O}_{C \cap D})_p = \mathcal{O}_X/(f, g)$ so $C \cdot D = \dim H^0(X, \mathcal{O}_{C \cap D})$. Recall for short exact sequences of sheaves on X, $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, taking right-derived functors of Γ (global section functor) we get an associated long exact sequence

$$(1) \qquad 0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to \cdots$$

where the H^i measure non-exactness.

Theorem 1. *If $\mathcal{F} = \tilde{M}$ on $X = \operatorname{Spec} A$ affine, $H^0(X, \mathcal{F}) = M, H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Theorem 2 (Serre)**.** *If X is a projective scheme over A, $\mathcal{O}_X(1)$ a very ample line bundle on X over A, and $\mathcal{F}$ a coherent sheaf on X, then each $H^i(X, \mathcal{F})$ is a f.g. A-module, and $\exists n_0$ dependent on $\mathcal{F}$ s.t. $\forall i > 0, n \geq n_0, H^i(X, \mathcal{F}(n)) = 0$.*

Theorem 3 (Grothendieck)**.** *Let X be a smooth, Noetherian scheme of dimension n. Then for $\mathcal{F}$ a sheaf of abelian groups on X and $i > n$, $H^i(X, \mathcal{F}) = 0$.*

Definition 2. *For $\mathcal{F}$ a sheaf on X, $\chi(F) = \sum(-1)^i h^i(X, \mathcal{F})$ is the* Euler-Poincaré *characteristic of F. Note that it is naturally additive for short-exact sequences of sheaves.*

Now we return to the setting of surfaces.

Definition 3. *For $\mathcal{L}, \mathcal{M} \in \operatorname{Pic} X$ (the group of line bundles), let $\mathcal{L} \cdot \mathcal{M} = \chi(\mathcal{O}_X) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{M}^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{M}^{-1})$.*

Proposition 1. *$(\cdot)$ defined above is a symmetric, bilinear form on $\operatorname{Pic} X$ s.t. if C, D are distinct, irreducible curves on X, $\mathcal{O}_X(C) \cdot \mathcal{O}_X(D) = C \cdot D$. That is, we can extend $\cdot$ from $\operatorname{Div} X$ to $\operatorname{Pic} X$.*

Proof. Let $S \in H^0(X, \mathcal{O}_X(C)), t \in H^0(X, \mathcal{O}_X(D))$ be nonzero sections. We have

$$(2) \quad 0 \to \mathcal{O}_X(-C-D) \overset{(t,-s)}{\to} \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-D) \overset{(s,t)}{\to} \mathcal{O}_X \to \mathcal{O}_{C \cap D} \to 0$$

is exact (by checking locally). Then the additivity of $\chi()$ gives us $\mathcal{O}_X(C) \cdot \mathcal{O}_X(D) = \chi(\mathcal{O}_{C \cap D}) = H^0(\mathcal{O}_{C \cap D}) = C \cdot D$. Symmetry is obvious for $(\cdot)$: we need to check that it is bilinear.

Lemma 1. *Let C be a nonsingular irreducible curve on X. For $L \in \operatorname{Pic} X$, we have $\mathcal{O}_X(C) \cdot L = \deg L|_C$.*

Proof. Tensoring $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ with L^{-1} gives

$$(3) \quad\quad\quad\quad 0 \to L^{-1}(-C) \to L^{-1} \to L^{-1} \otimes \mathcal{O}_C \to 0$$

Taking Euler characteristics of both sequences gives $\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) = \chi(\mathcal{O}_C)$ and $-\chi(L^{-1} + \chi(L^{-1}(-C)) = -\chi(L^{-1}|_C)$. Thus,

$$(4) \quad\quad\quad \mathcal{O}_X(C) \cdot L = \chi(\mathcal{O}_C) - \chi(L^{-1}|_C) = \deg L^{-1}|_C$$

by Riemann-Roch on C. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark. Riemann-Roch for curves says that, if C is a smooth irreducible curve and L is a line bundle on C, then $\chi(L) = \chi(\mathcal{O}_C) + \deg L$, or $h^0(L) - h^1(L) = \deg L + 1 - g$.

Now, for $L_1, L_2, L_3 \in \operatorname{Pic} X$, let $s(L_1, L_2, L_3) = L_1 \cdot (L_2 \otimes L_3) - L_1 \cdot L_2 - L_1 \cdot L_3$. The definition of $\cdot$ shows that this is symmetric in L_1, L_2, L_3, and the lemma shows that $s(L_1, L_2, L_3)$ is 0 when $L_1 = \mathcal{O}_X(C)$ for C a nonsingular curve. In general, let M be any invertible sheaf, and $\mathcal{O}(1)$ a very ample line bundle on X. Then $\exists n$ s.t. $M \otimes \mathcal{O}(n) = M(n)$ is generated by global sections. Then $M \otimes \mathcal{O}(n+1)$ and $\mathcal{O}(n+1$ are both very ample. By Bertini's theorem, we can write $M = \mathcal{O}_X(A - B)$, where A and B are nonsingular irreducible curves. Then $s(L, M, \mathcal{O}_X(B)) = 0$ (since s is symmetric in its 3 arguments), so that

$$(5) \qquad L \cdot M - L \cdot \mathcal{O}_X(A) - L \cdot \mathcal{O}_X(B) = 0$$

This shows that $L \cdot M$ is linear in L, since $L \cdot \mathcal{O}_X(A) = \deg L|_A$ and $L \cdot \mathcal{O}_X(B) = \deg L|_B$ are both linear in L. Similarly, $L \cdot M$ is linear in M, giving the desired bilinearity. $\qquad \square$