Last homework due Dec. 2, take home final due Dec. 10. Make-up lecture Dec. 11.
11/20/03
Let X be a variety. We had from last time a map
$O_X \xrightarrow{d} \Omega^1_X$, a sheaf of O_X-modules, where d was a derivation.

For all affine $U \subset X, R = \Gamma(U, O_X)$, then there was a map $R \xrightarrow{d} \Omega^1_X(U)$ was the universal derivation.

A derivation is a map $\delta : R \to M$ such that δ is k-linear and $\delta(fg) = f\delta(g) + g\delta(f)$. d is universal such that $R \xrightarrow{d} \Omega^1_X(U)$ factors any $\delta : R \to M$ with a unique R-linear map from $\Omega^1_X(U) \to M$.

Let $R = k[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$. Then $\Omega = Rdx_1 \oplus \ldots \oplus Rdx_n/(df_i)$.

That is, Ω is the cokernel of the map $R^r \xrightarrow{J} Rdx_1 \oplus \ldots \oplus Rdx_n$ where J is the jacobian

$$\begin{pmatrix} \delta f_1/\delta x_1 & \ldots & \delta f_r/\delta x_1 \\ \vdots & & \vdots \\ \delta f_1/\delta x_n \ldots \delta f_r/\delta x_n & & \end{pmatrix}$$

where $e_i \mapsto (\sum_j \frac{\delta f_i}{\delta x_j} dx_j)$.

Then $\Omega^1_X(D(f)) = \Omega \otimes_R R_f$. If $x \in X$ then the stalk $\Omega^1_{X,x} = \Omega \otimes_R R_{m_x}$.

We write $\Omega^1_X(x)$ to be $\Omega^1_{X,x} \otimes_{O_{X,x}} k$ from $\Omega^1_{X,x} \otimes_R k$ via the quotient map $R \to R_{m_x}$.

Example. $V(y^2 - x^2(x+1)) \subset \mathbb{A}^2$.

$$R \stackrel{(-3x^2-2x,2y)}{\longrightarrow} Rdx \oplus Rdy \to \Omega.$$

If we have $x = (a, b)$, then we have the sequence

$$k \stackrel{(-3a^2-2a,2b)}{\longrightarrow} kdx \oplus kdy \to \Omega^1_X(x) \to 0$$

by tensoring with k. Then $\dim \Omega^1_X(x) = 2$ if $x = (0,0)$, and 1 if $x \neq (0,0)$.

Lemma. Let X be a variety, $x \in X$. Then there is a natural isomorphism $\mathrm{Hom}_{O_{X,x}}(\Omega^1_{X,x}, k) = \mathrm{Hom}_k(\Omega^1_X(x), k) = \mathrm{Hom}_k(m_x/m_x^2, k)$. This last equality is what we must prove; the first is really just from properties of tensor products.

Cor. $O_{X,x}$ is regular $\iff \dim_k \Omega^1_X(x) = \dim X$.

Pf. of Lemma. $\mathrm{Hom}_{O_{X,x}}(\Omega^1_{X,x}, k)$ is the set of derivations $\delta : O_{X,x} \to k$. This is cheating a little because we only know this for affine opens, and this statement is passing to the limit, but it's okay.

Now note: any such map must kill m_x^2, because $\delta(xy) = x\delta(y) + y\delta(x)$ so if both things are in m_x then both parts on the right are in m_x and are thus 0 in k. So our set of homomorphisms is just the set of linear maps $m/m^2 \to k$.

We have a map

$$m/m^2 \hookrightarrow O_{X,x}/m^2 \xrightarrow{\delta} k,$$

where the composition is k-linear. Note that for any $f \in O_{X,x}$, we have that $\delta(f) = \delta(f(x) + (f - f(x)) = \delta(f - f(x))$ where $f - f(x)$ is in the maximal ideal so it is determined by this map.

Cor. $\dim_k \Omega^1_X(x) = \dim_k m/m^2$.

Def. $x \in X$ is a *smooth point* if $O_{X,x}$ is regular. If x is not a smooth point is is called a *singular* point. Basically singular points correspond to points where the Jacobian is "wrong" (ie, has strange rank or something).

Def. The *tangent sheaf* of X is $\mathrm{coHom}_{O_X}(\Omega^1_X, O_X)$.

Aside. If $\mathcal{F}, \mathcal{G}$ are sheaves of O_X-modules, then $\mathrm{coHom}_{O_X}(\mathcal{F}, \mathcal{G})$ is the sheaf[1] mapping $U \mapsto \mathrm{Hom}_{O_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$.

So $T_{X,x} = \mathrm{Hom}_{O_{X,x}}(\Omega^1_{X,x}, O_{X,x})$.

1 Curves

Def. A *curve* is a variety of dimension 1.

This is rather abstract as we've done things.

Theorem. The functor from the category of complete smooth curves with non-constant morphisms to the category of finitely generated field extensions $k \to K$ of transcendence degree 1, with k-algebra morphisms is defined as follows.

$C \mapsto k(C)$. Clearly $k(C)$ has transcendence degree 1 since C has dimension 1 (this was our original definition of dimension).

(1) This functor is an equivalence of categories. Furthermore, (2) every complete smooth curve is projective.

Sketch of proof. If we have $p \in C$, what do we know about $O_{C,p}$? It's a DVR, thanks to smoothness and dimension 1. Also, it sits inside $k(C)$. This will give a bijection between the points of C and DVRs in $k(C)$. It turns out this captures everything: given a field, the curve will be the set of DVRs in the field.

Aside about Number Theory. If K is a number field, then O_K is the integral closure of $\mathbb{Z}$, then every prime in O_K corresponds to a valuation ring in K. For example, $K = \mathbb{Q}$, $O_K = \mathbb{Z}$, then for every $q \in K, v_p(q) = ord_p(q)$. If this makes any sense to anyone.

Def. If K is a field, G a totally ordered group, then a *valuation* is a map $v : K - \{0\} \to G$ such that $v(xy) = v(x) + v(y)$ and $v(x + y) \geq min\{v(x), v(y)\}$. For us, the key example will be $K = k(C), p \in C$, $v_p(f)$ is the order of the zero / pole at p of f. For instance, if $C = \mathbb{A}^1$, then $k(C) = k(t)$. If $f \in k(t)$ then $f = a_1 t^r + a_2 t^{r+1} + \ldots$ for some r, so $v(f) = r$.

$R_v = \{x : v(x) \geq 0\} \cup \{0\}$ and $m_v = \{x | v(x) > 0\}$; these make a local ring.

[1] not obvious that it is a sheaf

v is *discrete* if we can take $G = \mathbb{Z}$. In general, if C is a smooth curve, $p \in C$, then $v_p : k(C) \setminus \{0\} \to \mathbb{Z}$; $v_p : O_{C,p} \setminus \{0\} \to \mathbb{Z}$, where $v_p(f) = max_n\{f \in m_p^n\}$. $v_p(f/g) = v_p(f) - v_p(g)$.

Def. If K is a field, $A, B \subset K$ local rings, we say that B *dominates* A if $A \subset B$ and $m_A = A \cap m_B$.

Theorem. Valuation rings are exactly the maximal local rings in K with respect to domination. [fact from AM]. We do not prove this here.

Let $k \to K$ be a field extension of transcendence degree 1. Let C_K be the set of DVRs in K. Say $U \subset C_K$ is open if it is cofinite or if $U = \emptyset$. Define $O_{C_K}(U)$ to be $\cap_{v \in U} O_v$, where O_v refers to the DVR corresponding to the valuation v. We can think of elements of O_v as functions, as $f(v)$ is the image of f in $O_v/m_v = k$.

Now we need to show that (C_K, O_{C_K}) is a smooth curve.

Lemma. Say $f \in K$. Then $\{v \in C_K | f \notin O_v\}$ is a finite set.

Pf. We have a fact: $\{v \in C_K | f \notin O_v\} = \{v \in C_K | 1/f \in m_v\}$. We know that $f \notin k$ or this set would be empty and we'd be done. Write $g = 1/f$. Let B be the integral closure of $k[g] \subset K$. B corresponds to a smooth curve C because every local ring is regular (this is from propertie of Dedekind domains...)

If $g \in m_v$, we get $m_v \subset B \to O_v \twoheadrightarrow evalk$ which corresponds to points in $V(g) \subset C$, so it's finite.

This also shows (C_K, O_{C_K}) is a variety. Say $v \in C_K$. Choose $g \in O_v$ non-zero. This corresponds to a smooth curve C, $k(C) = K$, where $\{\iota | R_\iota = O_{C,p}, p \in C\}$ is an open set around v.

Lemma. Say Y is an affine variety, $P, Q \in Y$ and say $O_{Y,P} \subset O_{Y,Q} \subset k(Y)$. Then $P = Q$. [Proof ommitted]

Out of time.