18.725: EXERCISE SET 3

DUE TUESDAY SEPTEMBER 30

(1) Suppose X is a topological space. If $f : G \to F$ is a morphism of sheaves of groups on X, the *image of* f is defined to be the sheaf associated to the presheaf $U \mapsto \mathrm{Im}(G(U) \to F(U))$. The map f is said to be *surjective* if F is equal to the image of f.

(i) Give an example of a surjective map of abelian sheaves $f : G \to F$ for which the map $G(U) \to F(U)$ is not surjective for some open set U in X.

(ii) Show that a map of abelian sheaves $f : G \to F$ is surjective if and only if for every point $x \in X$ the map on stalks $G_x \to F_x$ is surjective.

(2) Let $f : X \to Y$ be a continuous map between topological spaces. For any sheaf F on X, we defined in class the sheaf f_*F on Y by the formula $f_*F(U) = F(f^{-1}(U))$. This defines a functor from the category of sheaves on X to the category of sheaves on Y. Show that this functor has a left adjoint. That is, there is a functor $G \mapsto f^{-1}G$ from the category of sheaves on Y to the category of sheaves on X so that for any F on X and G on Y there is a natural isomorphism

$$\mathrm{Hom}(f^{-1}G, F) \simeq \mathrm{Hom}(G, f_*F).$$

Hint: let $f^{-1}G$ to be the sheaf associated to the presheaf on X which to any U associates $\varinjlim G(V)$, where the limit is taken over open sets V of Y containing $f(U)$.

(3) Show that the Zariski topology on $\mathbb{A}^2$ is not the topology obtained by identifying $\mathbb{A}^2 \simeq \mathbb{A}^1 \times \mathbb{A}^1$ and taking the product topology.

(4) A morphism $f : X \to Y$ between irreducible affine algebraic sets is called *finite* if the ring $\Gamma(X, \mathcal{O}_X)$ is a finite $\Gamma(Y, \mathcal{O}_Y)$–module. Show that if f is a finite morphism, then for each $y \in Y$, the inverse image $f^{-1}(y)$ is a finite set.

(5) Give an example of a morphism $f : X \to Y$ between irreducible algebraic sets which has finite fibers but is not a finite morphism.

(6) Suppose k has characteristic p and let $X \subset \mathbb{A}^n$ be an affine variety. Let $\sigma : k \to k$ denote the Frobenius automorphism of k and let $X^{(p)} \subset \mathbb{A}^n$ denote the algebraic set defined by the ideal

$$A = \{g \in k[X_1, \ldots, X_n] | g^\sigma \in I(X)\},$$

where if $g = \sum_{\underline{i}} a_{\underline{i}} X^{\underline{i}}$ we write g^σ for the polynomial $g = \sum_{\underline{i}} \sigma(a_{\underline{i}}) X^{\underline{i}}$. Show that Frobenius induces a natural morphism of affine k–varieties $X \to X^{(p)}$ which is a homeomorphism on the underlying topological spaces.

(7) Let $X = \mathbb{A}^2 - \{(0,0)\}$, and view X as a topological space with a sheaf of rings $\mathcal{O}_X$ by restricting the topology and sheaf on $\mathbb{A}^2$. Show that $(X, \mathcal{O}_X)$ is not an affine variety.

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(8) If $\mathcal{C}$ is a category and $A, B \in Ob(\mathcal{C})$ are objects, then the product $A \times B$ (if it exists) is defined to be an object $C \in Ob(\mathcal{C})$ together with maps $p_1 : C \to A$ and $p_2 : C \to B$ such that for any object $D \in Ob(\mathcal{C})$ the induced map

$$\mathrm{Hom}(D, C) \longrightarrow \mathrm{Hom}(D, A) \times \mathrm{Hom}(D, B)$$

is a bijection.

(i) Show that the product (C, p_1, p_2) of A and B in $\mathcal{C}$ is unique up to unique isomorphism if it exists.

(ii) Show that products exists in the category of affine varieties. Hint: first consider the case of $\mathbb{A}^n$ and $\mathbb{A}^m$.

(9) An *affine group variety* is a variety G together with a morphism $\mu : G \times G \to G$ such that the resulting operation on the points of G makes the points of G a group, and such that the inverse map $G \to G$ is a morphism.

(i) Show that $\mathbb{A}^1$ with $\mu : \mathbb{A}^2 \to \mathbb{A}^1$ given by $(a, b) \mapsto a + b$ is an affine group variety. We usually denote this group variety by $\mathbb{G}_a$.

(ii) If G is a group variety and X any affine variety, show that the set $\mathrm{Hom}(X, G)$ has a natural group structure.

(iii) If X is any affine variety, show that $\mathrm{Hom}(X, \mathbb{G}_a)$ is naturally isomorphic as a group to $\Gamma(X, \mathcal{O}_X)$ under addition.