2 General results of representation theory

2.1 Subrepresentations in semisimple representations

Let A be an algebra.

Definition 2.1. A semisimple (or completely reducible) representation of A is a direct sum of irreducible representations.

Example. Let V be an irreducible representation of A of dimension n. Then $Y = \mathrm{End}(V)$, with action of A by left multiplication, is a semisimple representation of A, isomorphic to nV (the direct sum of n copies of V). Indeed, any basis $v_1, ..., v_n$ of V gives rise to an isomorphism of representations $\mathrm{End}(V) \to nV$, given by $x \to (xv_1, ..., xv_n)$.

Remark. Note that by Schur's lemma, any semisimple representation V of A is canonically identified with $\oplus_X \mathrm{Hom}_A(X, V) \otimes X$, where X runs over all irreducible representations of A. Indeed, we have a natural map $f : \oplus_X \mathrm{Hom}(X, V) \otimes X \to V$, given by $g \otimes x \to g(x)$, $x \in X$, $g \in \mathrm{Hom}(X, V)$, and it is easy to verify that this map is an isomorphism.

We'll see now how Schur's lemma allows us to classify subrepresentations in finite dimensional semisimple representations.

Proposition 2.2. *Let $V_i, 1 \leq i \leq m$ be irreducible finite dimensional pairwise nonisomorphic representations of A, and W be a subrepresentation of $V = \oplus_{i=1}^m n_i V_i$. Then W is isomorphic to $\oplus_{i=1}^m r_i V_i$, $r_i \leq n_i$, and the inclusion $\phi : W \to V$ is a direct sum of inclusions $\phi_i : r_i V_i \to n_i V_i$ given by multiplication of a row vector of elements of V_i (of length r_i) by a certain r_i-by-n_i matrix X_i with linearly independent rows: $\phi(v_1, ..., v_{r_i}) = (v_1, ..., v_{r_i}) X_i$.*

Proof. The proof is by induction in $n := \sum_{i=1}^m n_i$. The base of induction ($n = 1$) is clear. To perform the induction step, let us assume that W is nonzero, and fix an irreducible subrepresentation $P \subset W$. Such P exists (Problem 1.20). [2] Now, by Schur's lemma, P is isomorphic to V_i for some i, and the inclusion $\phi|_P : P \to V$ factors through $n_i V_i$, and upon identification of P with V_i is given by the formula $v \mapsto (vq_1, ..., vq_{n_i})$, where $q_l \in k$ are not all zero.

Now note that the group $G_i = GL_{n_i}(k)$ of invertible n_i-by-n_i matrices over k acts on $n_i V_i$ by $(v_1, ..., v_{n_i}) \to (v_1, ..., v_{n_i}) g_i$ (and by the identity on $n_j V_j$, $j = i$), and therefore acts on the set of subrepresentations of V, preserving the property we need to establish: namely, under the action of g_i, the matrix X_i goes to $X_i g_i$, while $X_j, j = i$ don't change. Take $g_i \in G_i$ such that $(q_1, ..., q_{n_i}) g_i = (1, 0, ..., 0)$. Then $W g_i$ contains the first summand V_i of $n_i V_i$ (namely, it is $P g_i$), hence $W g_i = V_i \oplus W'$, where $W' \subset n_1 V_1 \oplus ... \oplus (n_i - 1) V_i \oplus ... \oplus n_m V_m$ is the kernel of the projection of $W g_i$ to the first summand V_i along the other summands. Thus the required statement follows from the induction assumption. $\square$

Remark 2.3. In Proposition 2.2, it is not important that k is algebraically closed, nor it matters that V is finite dimensional. If these assumptions are dropped, the only change needed is that the entries of the matrix X_i are no longer in k but in $D_i = \mathrm{End}_A(V_i)$, which is, as we know, a division algebra. The proof of this generalized version of Proposition 2.2 is the same as before (check it!).

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[2] Another proof of the existence of P, which does not use the finite dimensionality of V, is by induction in n. Namely, if W itself is not irreducible, let K be the kernel of the projection of W to the first summand V_1. Then K is a subrepresentation of $(n_1 - 1)V_1 \oplus ... \oplus n_m V_m$, which is nonzero since W is not irreducible, so K contains an irreducible subrepresentation by the induction assumption.

2.2 The density theorem

Let A be an algebra over an algebraically closed field k.

Corollary 2.4. *Let V be an irreducible finite dimensional representation of A, and $v_1, ..., v_n \in V$ be any linearly independent vectors. Then for any $w_1, ..., w_n \in V$ there exists an element $a \in A$ such that $av_i = w_i$.*

Proof. Assume the contrary. Then the image of the map $A \to nV$ given by $a \to (av_1, ..., av_n)$ is a proper subrepresentation, so by Proposition 2.2 it corresponds to an r-by-n matrix X, $r < n$. Thus, taking $a = 1$, we see that there exist vectors $u_1, ..., u_r \in V$ such that $(u_1, ..., u_r)X = (v_1, ..., v_n)$. Let $(q_1, ..., q_n)$ be a nonzero vector such that $X(q_1, ..., q_n)^T = 0$ (it exists because $r < n$). Then $\sum q_i v_i = (u_1, ..., u_r)X(q_1, ..., q_n)^T = 0$, i.e. $\sum q_i v_i = 0$ - a contradiction with the linear independence of v_i. ☐

Theorem 2.5. *(the Density Theorem). (i) Let V be an irreducible finite dimensional representation of A. Then the map $\rho : A \to \mathrm{End}V$ is surjective.*

(ii) Let $V = V_1 \oplus ... \oplus V_r$, where V_i are irreducible pairwise nonisomorphic finite dimensional representations of A. Then the map $\oplus_{i=1}^r \rho_i : A \to \oplus_{i=1}^r \mathrm{End}(V_i)$ is surjective.

Proof. (i) Let B be the image of A in $\mathrm{End}(V)$. We want to show that $B = \mathrm{End}(V)$. Let $c \in \mathrm{End}(V)$, $v_1, ..., v_n$ be a basis of V, and $w_i = cv_i$. By Corollary 2.4, there exists $a \in A$ such that $av_i = w_i$. Then a maps to c, so $c \in B$, and we are done.

(ii) Let B_i be the image of A in $\mathrm{End}(V_i)$, and B be the image of A in $\oplus_{i=1}^r \mathrm{End}(V_i)$. Recall that as a representation of A, $\oplus_{i=1}^r \mathrm{End}(V_i)$ is semisimple: it is isomorphic to $\oplus_{i=1}^r d_i V_i$, where $d_i = \dim V_i$. Then by Proposition 2.2, $B = \oplus_i B_i$. On the other hand, (i) implies that $B_i = \mathrm{End}(V_i)$. Thus (ii) follows. ☐

2.3 Representations of direct sums of matrix algebras

In this section we consider representations of algebras $A = \bigoplus_i \mathrm{Mat}_{d_i}(k)$ for any field k.

Theorem 2.6. *Let $A = \bigoplus_{i=1}^r \mathrm{Mat}_{d_i}(k)$. Then the irreducible representations of A are $V_1 = k^{d_1}, \ldots, V_r = k^{d_r}$, and any finite dimensional representation of A is a direct sum of copies of $V_1, \ldots, V_r$.*

In order to prove Theorem 2.6, we shall need the notion of a dual representation.

Definition 2.7. (Dual representation) Let V be a representation of any algebra A. Then the dual representation V^* is the representation of the opposite algebra A^{op} (or, equivalently, right A-module) with the action

$$(f \cdot a)(v) := f(av).$$

Proof of Theorem 2.6. First, the given representations are clearly irreducible, as for any $v = 0, w \in V_i$, there exists $a \in A$ such that $av = w$. Next, let X be an n-dimensional representation of A. Then, X^* is an n-dimensional representation of A^{op}. But $(\mathrm{Mat}_{d_i}(k))^{\mathrm{op}} \cong \mathrm{Mat}_{d_i}(k)$ with isomorphism $\varphi(X) = X^T$, as $(BC)^T = C^T B^T$. Thus, $A \cong A^{\mathrm{op}}$ and X^* may be viewed as an n-dimensional representation of A. Define

$$\phi : \underbrace{A \oplus \cdots \oplus A}_{n \text{ copies}} \longrightarrow X^*$$

by

$$\phi(a_1, \ldots, a_n) = a_1 y_1 + \cdots + a_n y_n$$

where $\{y_i\}$ is a basis of X^*. ϕ is clearly surjective, as $k \subset A$. Thus, the dual map $\phi^* : X \longrightarrow A^{n*}$ is injective. But $A^{n*} \cong A^n$ as representations of A (check it!). Hence, $\operatorname{Im} \phi^* \cong X$ is a subrepresentation of A^n. Next, $\mathrm{Mat}_{d_i}(k) = d_i V_i$, so $A = \oplus_{i=1}^r d_i V_i$, $A^n = \oplus_{i=1}^r n d_i V_i$, as a representation of A. Hence by Proposition 2.2, $X = \oplus_{i=1}^r m_i V_i$, as desired. $\square$

Exercise. The goal of this exercise is to give an alternative proof of Theorem 2.6, not using any of the previous results of Chapter 2.

Let A_1, A_2, ..., A_n be n algebras with units 1_1, 1_2, ..., 1_n, respectively. Let $A = A_1 \oplus A_2 \oplus ... \oplus A_n$. Clearly, $1_i 1_j = \delta_{ij} 1_i$, and the unit of A is $1 = 1_1 + 1_2 + ... + 1_n$.

For every representation V of A, it is easy to see that $1_i V$ is a representation of A_i for every $i \in \{1, 2, ..., n\}$. Conversely, if V_1, V_2, ..., V_n are representations of A_1, A_2, ..., A_n, respectively, then $V_1 \oplus V_2 \oplus ... \oplus V_n$ canonically becomes a representation of A (with $(a_1, a_2, ..., a_n) \in A$ acting on $V_1 \oplus V_2 \oplus ... \oplus V_n$ as $(v_1, v_2, ..., v_n) \mapsto (a_1 v_1, a_2 v_2, ..., a_n v_n)$).

(a) Show that a representation V of A is irreducible if and only if $1_i V$ is an irreducible representation of A_i for exactly one $i \in \{1, 2, ..., n\}$, while $1_i V = 0$ for all the other i. Thus, classify the irreducible representations of A in terms of those of A_1, A_2, ..., A_n.

(b) Let $d \in \mathbb{N}$. Show that the only irreducible representation of $\mathrm{Mat}_d(k)$ is k^d, and every finite dimensional representation of $\mathrm{Mat}_d(k)$ is a direct sum of copies of k^d.

Hint: For every $(i, j) \in \{1, 2, ..., d\}^2$, let $E_{ij} \in \mathrm{Mat}_d(k)$ be the matrix with 1 in the ith row of the jth column and 0's everywhere else. Let V be a finite dimensional representation of $\mathrm{Mat}_d(k)$. Show that $V = E_{11} V \oplus E_{22} V \oplus ... \oplus E_{dd} V$, and that $\Phi_i : E_{11} V \to E_{ii} V$, $v \mapsto E_{i1} v$ is an isomorphism for every $i \in \{1, 2, ..., d\}$. For every $v \in E_{11} V$, denote $S(v) = \langle E_{11} v, E_{21} v, ..., E_{d1} v \rangle$. Prove that $S(v)$ is a subrepresentation of V isomorphic to k^d (as a representation of $\mathrm{Mat}_d(k)$), and that $v \in S(v)$. Conclude that $V = S(v_1) \oplus S(v_2) \oplus ... \oplus S(v_k)$, where $\{v_1, v_2, ..., v_k\}$ is a basis of $E_{11} V$.

(c) Conclude Theorem 2.6.

2.4 Filtrations

Let A be an algebra. Let V be a representation of A. A (finite) *filtration* of V is a sequence of subrepresentations $0 = V_0 \subset V_1 \subset ... \subset V_n = V$.

Lemma 2.8. *Any finite dimensional representation V of an algebra A admits a finite filtration $0 = V_0 \subset V_1 \subset ... \subset V_n = V$ such that the successive quotients V_i/V_{i-1} are irreducible.*

Proof. The proof is by induction in $\dim(V)$. The base is clear, and only the induction step needs to be justified. Pick an irreducible subrepresentation $V_1 \subset V$, and consider the representation $U = V/V_1$. Then by the induction assumption U has a filtration $0 = U_0 \subset U_1 \subset ... \subset U_{n-1} = U$ such that U_i/U_{i-1} are irreducible. Define V_i for $i \geq 2$ to be the preimages of U_{i-1} under the tautological projection $V \to V/V_1 = U$. Then $0 = V_0 \subset V_1 \subset V_2 \subset ... \subset V_n = V$ is a filtration of V with the desired property. $\square$

2.5 Finite dimensional algebras

Definition 2.9. The **radical** of a finite dimensional algebra A is the set of all elements of A which act by 0 in all irreducible representations of A. It is denoted $\mathrm{Rad}(A)$.

Proposition 2.10. *$Rad(A)$ is a two-sided ideal.*

Proof. Easy. □

Proposition 2.11. *Let A be a finite dimensional algebra.*

(i) Let I be a nilpotent two-sided ideal in A, i.e., $I^n = 0$ for some n. Then $I \subset Rad(A)$.

(ii) $Rad(A)$ is a nilpotent ideal. Thus, $Rad(A)$ is the largest nilpotent two-sided ideal in A.

Proof. (i) Let V be an irreducible representation of A. Let $v \in V$. Then $Iv \subset V$ is a subrepresentation. If $Iv = 0$ then $Iv = V$ so there is $x \in I$ such that $xv = v$. Then $x^n = 0$, a contradiction. Thus $Iv = 0$, so I acts by 0 in V and hence $I \subset \mathrm{Rad}(A)$.

(ii) Let $0 = A_0 \subset A_1 \subset ... \subset A_n = A$ be a filtration of the regular representation of A by subrepresentations such that A_{i+1}/A_i are irreducible. It exists by Lemma 2.8. Let $x \in \mathrm{Rad}(A)$. Then x acts on A_{i+1}/A_i by zero, so x maps A_{i+1} to A_i. This implies that $\mathrm{Rad}(A)^n = 0$, as desired. □

Theorem 2.12. *A finite dimensional algebra A has only finitely many irreducible representations V_i up to isomorphism, these representations are finite dimensional, and*

$$A/Rad(A) \cong \bigoplus_i \mathrm{End}\, V_i.$$

Proof. First, for any irreducible representation V of A, and for any nonzero $v \in V$, $Av \subseteq V$ is a finite dimensional subrepresentation of V. (It is finite dimensional as A is finite dimensional.) As V is irreducible and $Av = 0$, $V = Av$ and V is finite dimensional.

Next, suppose we have non-isomorphic irreducible representations $V_1, V_2, \ldots, V_r$. By Theorem 2.5, the homomorphism

$$\bigoplus_i \rho_i : A \longrightarrow \bigoplus_i \mathrm{End}\, V_i$$

is surjective. So $r \leq \sum_i \dim \mathrm{End}\, V_i \leq \dim A$. Thus, A has only finitely many non-isomorphic irreducible representations (at most $\dim A$).

Now, let $V_1, V_2, \ldots, V_r$ be all non-isomorphic irreducible finite dimensional representations of A. By Theorem 2.5, the homomorphism

$$\bigoplus_i \rho_i : A \longrightarrow \bigoplus_i \mathrm{End}\, V_i$$

is surjective. The kernel of this map, by definition, is exactly $\mathrm{Rad}(A)$. □

Corollary 2.13. $\sum_i (\dim V_i)^2 \leq \dim A$, *where the V_i's are the irreducible representations of A.*

Proof. As $\dim \mathrm{End}\, V_i = (\dim V_i)^2$, Theorem 2.12 implies that $\dim A - \dim \mathrm{Rad}(A) = \sum_i \dim \mathrm{End}\, V_i = \sum_i (\dim V_i)^2$. As $\dim \mathrm{Rad}(A) \geq 0$, $\sum_i (\dim V_i)^2 \leq \dim A$. □

Example 2.14. 1. Let $A = k[x]/(x^n)$. This algebra has a unique irreducible representation, which is a 1-dimensional space k, in which x acts by zero. So the radical $\mathrm{Rad}(A)$ is the ideal (x).

2. Let A be the algebra of upper triangular n by n matrices. It is easy to check that the irreducible representations of A are V_i, $i = 1, ..., n$, which are 1-dimensional, and any matrix x acts by x_{ii}. So the radical $\mathrm{Rad}(A)$ is the ideal of strictly upper triangular matrices (as it is a nilpotent ideal and contains the radical). A similar result holds for block-triangular matrices.

Definition 2.15. A finite dimensional algebra A is said to be **semisimple** if $\mathrm{Rad}(A) = 0$.

Proposition 2.16. *For a finite dimensional algebra A, the following are equivalent:*

1. *A is semisimple.*
2. *$\sum_i (\dim V_i)^2 = \dim A$, where the V_i's are the irreducible representations of A.*
3. *$A \cong \bigoplus_i \mathrm{Mat}_{d_i}(k)$ for some d_i.*
4. *Any finite dimensional representation of A is completely reducible (that is, isomorphic to a direct sum of irreducible representations).*
5. *A is a completely reducible representation of A.*

Proof. As $\dim A - \dim \mathrm{Rad}(A) = \sum_i (\dim V_i)^2$, clearly $\dim A = \sum_i (\dim V_i)^2$ if and only if $\mathrm{Rad}(A) = 0$. Thus, (1) $\Leftrightarrow$ (2).

Next, by Theorem 2.12, if $\mathrm{Rad}(A) = 0$, then clearly $A \cong \bigoplus_i \mathrm{Mat}_{d_i}(k)$ for $d_i = \dim V_i$. Thus, (1) $\Rightarrow$ (3). Conversely, if $A \cong \bigoplus_i \mathrm{Mat}_{d_i}(k)$, then by Theorem 2.6, $\mathrm{Rad}(A) = 0$, so A is semisimple. Thus (3) $\Rightarrow$ (1).

Next, (3) $\Rightarrow$ (4) by Theorem 2.6. Clearly (4) $\Rightarrow$ (5). To see that (5) $\Rightarrow$ (3), let $A = \bigoplus_i n_i V_i$. Consider $\mathrm{End}_A(A)$ (endomorphisms of A as a representation of A). As the V_i's are pairwise non-isomorphic, by Schur's lemma, no copy of V_i in A can be mapped to a distinct V_j. Also, again by Schur's lemma, $\mathrm{End}_A(V_i) = k$. Thus, $\mathrm{End}_A(A) \cong \bigoplus_i \mathrm{Mat}_{n_i}(k)$. But $\mathrm{End}_A(A) \cong A^{\mathrm{op}}$ by Problem 1.22, so $A^{\mathrm{op}} \cong \bigoplus_i \mathrm{Mat}_{n_i}(k)$. Thus, $A \cong (\bigoplus_i \mathrm{Mat}_{n_i}(k))^{\mathrm{op}} = \bigoplus_i \mathrm{Mat}_{n_i}(k)$, as desired. $\square$

2.6 Characters of representations

Let A be an algebra and V a finite-dimensional representation of A with action ρ. Then the *character* of V is the linear function $\chi_V : A \to k$ given by

$$\chi_V(a) = \mathrm{tr}|_V(\rho(a)).$$

If $[A, A]$ is the span of commutators $[x, y] := xy - yx$ over all $x, y \in A$, then $[A, A] \subseteq \ker \chi_V$. Thus, we may view the character as a mapping $\chi_V : A/[A, A] \to k$.

Exercise. Show that if $W \subset V$ are finite dimensional representations of A, then $\chi_V = \chi_W + \chi_{V/W}$.

Theorem 2.17. *(i) Characters of (distinct) irreducible finite-dimensional representations of A are linearly independent.*

(ii) If A is a finite-dimensional semisimple algebra, then these characters form a basis of $(A/[A, A])^*$.

Proof. (i) If $V_1, \dots, V_r$ are nonisomorphic irreducible finite-dimensional representations of A, then $\rho_{V_1} \oplus \cdots \oplus \rho_{V_r} : A \to \text{End } V_1 \oplus \cdots \oplus \text{End } V_r$ is surjective by the density theorem, so $\chi_{V_1}, \dots, \chi_{V_r}$ are linearly independent. (Indeed, if $\sum \lambda_i \chi_{V_i}(a) = 0$ for all $a \in A$, then $\sum \lambda_i \text{Tr}(M_i) = 0$ for all $M_i \in \text{End}_k V_i$. But each $\text{tr}(M_i)$ can range independently over k, so it must be that $\lambda_1 = \cdots = \lambda_r = 0$.)

(ii) First we prove that $[\text{Mat}_d(k), \text{Mat}_d(k)] = sl_d(k)$, the set of all matrices with trace 0. It is clear that $[\text{Mat}_d(k), \text{Mat}_d(k)] \subseteq sl_d(k)$. If we denote by E_{ij} the matrix with 1 in the ith row of the jth column and 0's everywhere else, we have $[E_{ij}, E_{jm}] = E_{im}$ for $i = m$, and $[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1}$. Now $\{E_{im}\} \cup \{E_{ii} - E_{i+1,i+1}\}$ forms a basis in $sl_d(k)$, so indeed $[\text{Mat}_d(k), \text{Mat}_d(k)] = sl_d(k)$, as claimed.

By semisimplicity, we can write $A = \text{Mat}_{d_1}(k) \oplus \cdots \oplus \text{Mat}_{d_r}(k)$. Then $[A, A] = sl_{d_1}(k) \oplus \cdots \oplus sl_{d_r}(k)$, and $A/[A, A] \cong k^r$. By Theorem 2.6, there are exactly r irreducible representations of A (isomorphic to $k^{d_1}, \dots, k^{d_r}$, respectively), and therefore r linearly independent characters on the r-dimensional vector space $A/[A, A]$. Thus, the characters form a basis. □

2.7 The Jordan-Hölder theorem

We will now state and prove two important theorems about representations of finite dimensional algebras - the Jordan-Hölder theorem and the Krull-Schmidt theorem.

Theorem 2.18. *(Jordan-Hölder theorem). Let V be a finite dimensional representation of A, and $0 = V_0 \subset V_1 \subset \dots \subset V_n = V$, $0 = V_0' \subset \dots \subset V_m' = V$ be filtrations of V, such that the representations $W_i := V_i/V_{i-1}$ and $W_i' := V_i'/V_{i-1}'$ are irreducible for all i. Then $n = m$, and there exists a permutation σ of $1, \dots, n$ such that $W_{\sigma(i)}$ is isomorphic to W_i'.*

Proof. **First proof** (for k of characteristic zero). The character of V obviously equals the sum of characters of W_i, and also the sum of characters of W_i'. But by Theorem 2.17, the characters of irreducible representations are linearly independent, so the multiplicity of every irreducible representation W of A among W_i and among W_i' are the same. This implies the theorem. [3]

Second proof (general). The proof is by induction on $\dim V$. The base of induction is clear, so let us prove the induction step. If $W_1 = W_1'$ (as subspaces), we are done, since by the induction assumption the theorem holds for V/W_1. So assume $W_1 = W_1'$. In this case $W_1 \cap W_1' = 0$ (as W_1, W_1' are irreducible), so we have an embedding $f : W_1 \oplus W_1' \to V$. Let $U = V/(W_1 \oplus W_1')$, and $0 = U_0 \subset U_1 \subset \dots \subset U_p = U$ be a filtration of U with simple quotients $Z_i = U_i/U_{i-1}$ (it exists by Lemma 2.8). Then we see that:

1) V/W_1 has a filtration with successive quotients $W_1', Z_1, \dots, Z_p$, and another filtration with successive quotients $W_2, \dots., W_n$.

2) V/W_1' has a filtration with successive quotients $W_1, Z_1, \dots, Z_p$, and another filtration with successive quotients $W_2', \dots., W_n'$.

By the induction assumption, this means that the collection of irreducible representations with multiplicities $W_1, W_1', Z_1, \dots, Z_p$ coincides on one hand with $W_1, \dots, W_n$, and on the other hand, with $W_1', \dots, W_m'$. We are done. □

The Jordan-Hölder theorem shows that the number n of terms in a filtration of V with irreducible successive quotients does not depend on the choice of a filtration, and depends only on

[3] This proof does not work in characteristic p because it only implies that the multiplicities of W_i and W_i' are the same modulo p, which is not sufficient. In fact, the character of the representation pV, where V is any representation, is zero.

V. This number is called the *length* of V. It is easy to see that n is also the maximal length of a filtration of V in which all the inclusions are strict.

The sequence of the irreducible representations $W_1, ..., W_n$ enumerated in the order they appear from some filtration of V as successive quoteints is called a **Jordan-Hölder series** of V.

2.8 The Krull-Schmidt theorem

Theorem 2.19. *(Krull-Schmidt theorem) Any finite dimensional representation of A can be uniquely (up to an isomorphism and order of summands) decomposed into a direct sum of indecomposable representations.*

Proof. It is clear that a decomposition of V into a direct sum of indecomposable representations exists, so we just need to prove uniqueness. We will prove it by induction on $\dim V$. Let $V = V_1 \oplus ... \oplus V_m = V_1' \oplus ... \oplus V_n'$. Let $i_s : V_s \to V$, $i_s' : V_s' \to V$, $p_s : V \to V_s$, $p_s' : V \to V_s'$ be the natural maps associated to these decompositions. Let $\theta_s = p_1 i_s' p_s' i_1 : V_1 \to V_1$. We have $\sum_{s=1}^n \theta_s = 1$. Now we need the following lemma.

Lemma 2.20. *Let W be a finite dimensional indecomposable representation of A. Then*

(i) Any homomorphism $\theta : W \to W$ is either an isomorphism or nilpotent;

(ii) If $\theta_s : W \to W$, $s = 1, ..., n$ are nilpotent homomorphisms, then so is $\theta := \theta_1 + ... + \theta_n$.

Proof. (i) Generalized eigenspaces of θ are subrepresentations of W, and W is their direct sum. Thus, θ can have only one eigenvalue λ. If λ is zero, θ is nilpotent, otherwise it is an isomorphism.

(ii) The proof is by induction in n. The base is clear. To make the induction step ($n-1$ to n), assume that θ is not nilpotent. Then by (i) θ is an isomorphism, so $\sum_{i=1}^n \theta^{-1}\theta_i = 1$. The morphisms $\theta^{-1}\theta_i$ are not isomorphisms, so they are nilpotent. Thus $1 - \theta^{-1}\theta_n = \theta^{-1}\theta_1 + ... + \theta^{-1}\theta_{n-1}$ is an isomorphism, which is a contradiction with the induction assumption. □

By the lemma, we find that for some s, θ_s must be an isomorphism; we may assume that $s = 1$. In this case, $V_1' = \mathrm{Im}(p_1' i_1) \oplus \mathrm{Ker}(p_1 i_1')$, so since V_1' is indecomposable, we get that $f := p_1' i_1 : V_1 \to V_1'$ and $g := p_1 i_1' : V_1' \to V_1$ are isomorphisms.

Let $B = \oplus_{j>1} V_j$, $B' = \oplus_{j>1} V_j'$; then we have $V = V_1 \oplus B = V_1' \oplus B'$. Consider the map $h : B \to B'$ defined as a composition of the natural maps $B \to V \to B'$ attached to these decompositions. We claim that h is an isomorphism. To show this, it suffices to show that $\mathrm{Ker} h = 0$ (as h is a map between spaces of the same dimension). Assume that $v \in \mathrm{Ker} h \subset B$. Then $v \in V_1'$. On the other hand, the projection of v to V_1 is zero, so $gv = 0$. Since g is an isomorphism, we get $v = 0$, as desired.

Now by the induction assumption, $m = n$, and $V_j \cong V'_{\sigma(j)}$ for some permutation σ of $2, ..., n$. The theorem is proved. □

Exercise. Let A be the algebra of real-valued continuous functions on $\mathbb{R}$ which are periodic with period 1. Let M be the A-module of continuous functions f on $\mathbb{R}$ which are antiperiodic with period 1, i.e., $f(x+1) = -f(x)$.

(i) Show that A and M are indecomposable A-modules.

(ii) Show that A is not isomorphic to M but $A \oplus A$ is isomorphic to $M \oplus M$.

Remark. Thus, we see that in general, the Krull-Schmidt theorem fails for infinite dimensional modules. However, it still holds for modules of *finite length*, i.e., modules M such that any filtration of M has length bounded above by a certain constant $l = l(M)$.

2.9 Problems

Problem 2.21. Extensions of representations. *Let A be an algebra, and V, W be a pair of representations of A. We would like to classify representations U of A such that V is a subrepresentation of U, and $U/V = W$. Of course, there is an obvious example $U = V \oplus W$, but are there any others?*

Suppose we have a representation U as above. As a vector space, it can be (non-uniquely) identified with $V \oplus W$, so that for any $a \in A$ the corresponding operator $\rho_U(a)$ has block triangular form

$$\rho_U(a) = \begin{pmatrix} \rho_V(a) & f(a) \\ 0 & \rho_W(a) \end{pmatrix},$$

where $f : A \to \mathrm{Hom}_k(W, V)$ is a linear map.

(a) What is the necessary and sufficient condition on $f(a)$ under which $\rho_U(a)$ is a representation? Maps f satisfying this condition are called (1-)cocycles (of A with coefficients in $\mathrm{Hom}_k(W, V)$). They form a vector space denoted $Z^1(W, V)$.

(b) Let $X : W \to V$ be a linear map. The coboundary of X, dX, is defined to be the function $A \to \mathrm{Hom}_k(W, V)$ given by $dX(a) = \rho_V(a)X - X\rho_W(a)$. Show that dX is a cocycle, which vanishes if and only if X is a homomorphism of representations. Thus coboundaries form a subspace $B^1(W, V) \subset Z^1(W, V)$, which is isomorphic to $\mathrm{Hom}_k(W, V)/\mathrm{Hom}_A(W, V)$. The quotient $Z^1(W, V)/B^1(W, V)$ is denoted $\mathrm{Ext}^1(W, V)$.

(c) Show that if $f, f' \in Z^1(W, V)$ and $f - f' \in B^1(W, V)$ then the corresponding extensions U, U' are isomorphic representations of A. Conversely, if $\phi : U \to U'$ is an isomorphism such that

$$\phi(a) = \begin{pmatrix} 1_V & * \\ 0 & 1_W \end{pmatrix}$$

then $f - f' \in B^1(V, W)$. Thus, the space $\mathrm{Ext}^1(W, V)$ "classifies" extensions of W by V.

(d) Assume that W, V are finite dimensional irreducible representations of A. For any $f \in Ext^1(W, V)$, let U_f be the corresponding extension. Show that U_f is isomorphic to $U_{f'}$ as representations if and only if f and f' are proportional. Thus isomorphism classes (as representations) of nontrivial extensions of W by V (i.e., those not isomorphic to $W \oplus V$) are parametrized by the projective space $\mathbb{P}\mathrm{Ext}^1(W, V)$. In particular, every extension is trivial if and only if $\mathrm{Ext}^1(W, V) = 0$.

Problem 2.22. *(a) Let $A = \mathbf{C}[x_1, ..., x_n]$, and V_a, V_b be one-dimensional representations in which x_i act by a_i and b_i, respectively ($a_i, b_i \in \mathbf{C}$). Find $\mathrm{Ext}^1(V_a, V_b)$ and classify 2-dimensional representations of A.*

(b) Let B be the algebra over $\mathbf{C}$ generated by $x_1, ..., x_n$ with the defining relations $x_i x_j = 0$ for all i, j. Show that for $n > 1$ the algebra B has infinitely many non-isomorphic indecomposable representations.

Problem 2.23. *Let Q be a quiver without oriented cycles, and P_Q the path algebra of Q. Find irreducible representations of P_Q and compute Ext^1 between them. Classify 2-dimensional representations of P_Q.*

Problem 2.24. *Let A be an algebra, and V a representation of A. Let $\rho: A \to \mathrm{End}V$. A formal deformation of V is a formal series*

$$\tilde{\rho} = \rho_0 + t\rho_1 + ... + t^n\rho_n + ...,$$

where $\rho_i : A \to \mathrm{End}(V)$ are linear maps, $\rho_0 = \rho$, and $\tilde{\rho}(ab) = \tilde{\rho}(a)\tilde{\rho}(b)$.

If $b(t) = 1 + b_1 t + b_2 t^2 + ...$, where $b_i \in \mathrm{End}(V)$, and $\tilde{\rho}$ is a formal deformation of ρ, then $b\tilde{\rho}b^{-1}$ is also a deformation of ρ, which is said to be isomorphic to $\tilde{\rho}$.

(a) Show that if $\mathrm{Ext}^1(V, V) = 0$, then any deformation of ρ is trivial, i.e., isomorphic to ρ.

(b) Is the converse to (a) true? (consider the algebra of dual numbers $A = k[x]/x^2$).

Problem 2.25. The Clifford algebra. *Let V be a finite dimensional complex vector space equipped with a symmetric bilinear form $(,)$. The Clifford algebra $\mathrm{Cl}(V)$ is the quotient of the tensor algebra TV by the ideal generated by the elements $v \otimes v - (v, v)1$, $v \in V$. More explicitly, if $x_i, 1 \le i \le N$ is a basis of V and $(x_i, x_j) = a_{ij}$ then $\mathrm{Cl}(V)$ is generated by x_i with defining relations*

$$x_i x_j + x_j x_i = 2a_{ij}, x_i^2 = a_{ii}.$$

Thus, if $(,) = 0$, $\mathrm{Cl}(V) = \wedge V$.

(i) Show that if $(,)$ is nondegenerate then $\mathrm{Cl}(V)$ is semisimple, and has one irreducible representation of dimension 2^n if $\dim V = 2n$ (so in this case $\mathrm{Cl}(V)$ is a matrix algebra), and two such representations if $\dim(V) = 2n+1$ (i.e., in this case $\mathrm{Cl}(V)$ is a direct sum of two matrix algebras).

Hint. In the even case, pick a basis $a_1, ..., a_n, b_1, ..., b_n$ of V in which $(a_i, a_j) = (b_i, b_j) = 0$, $(a_i, b_j) = \delta_{ij}/2$, and construct a representation of $\mathrm{Cl}(V)$ on $S := \wedge(a_1, ..., a_n)$ in which b_i acts as "differentiation" with respect to a_i. Show that S is irreducible. In the odd case the situation is similar, except there should be an additional basis vector c such that $(c, a_i) = (c, b_i) = 0$, $(c, c) = 1$, and the action of c on S may be defined either by $(-1)^{\mathrm{degree}}$ or by $(-1)^{\mathrm{degree}+1}$, giving two representations S_+, S_- (why are they non-isomorphic?). Show that there is no other irreducible representations by finding a spanning set of $\mathrm{Cl}(V)$ with $2^{\dim V}$ elements.

(ii) Show that $\mathrm{Cl}(V)$ is semisimple if and only if $(,)$ is nondegenerate. If $(,)$ is degenerate, what is $\mathrm{Cl}(V)/Rad(\mathrm{Cl}(V))$?

2.10 Representations of tensor products

Let A, B be algebras. Then $A \otimes B$ is also an algebra, with multiplication $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

Exercise. Show that $\mathrm{Mat}_m(k) \otimes \mathrm{Mat}_n(k) \cong \mathrm{Mat}_{mn}(k)$.

The following theorem describes irreducible finite dimensional representations of $A \otimes B$ in terms of irreducible finite dimensional representations of A and those of B.

Theorem 2.26. *(i) Let V be an irreducible finite dimensional representation of A and W an irreducible finite dimensional representation of B. Then $V \otimes W$ is an irreducible representation of $A \otimes B$.*

(ii) Any irreducible finite dimensional representation M of $A \otimes B$ has the form (i) for unique V and W.

Remark 2.27. Part (ii) of the theorem typically fails for infinite dimensional representations; e.g. it fails when A is the Weyl algebra in characteristic zero. Part (i) also may fail. E.g. let $A = B = V = W = \mathbb{C}(x)$. Then (i) fails, as $A \otimes B$ is not a field.

Proof. (i) By the density theorem, the maps $A \to \operatorname{End} V$ and $B \to \operatorname{End} W$ are surjective. Therefore, the map $A \otimes B \to \operatorname{End} V \otimes \operatorname{End} W = \operatorname{End}(V \otimes W)$ is surjective. Thus, $V \otimes W$ is irreducible.

(ii) First we show the existence of V and W. Let A', B' be the images of A, B in $\operatorname{End} M$. Then A', B' are finite dimensional algebras, and M is a representation of $A' \otimes B'$, so we may assume without loss of generality that A and B are finite dimensional.

In this case, we claim that $\operatorname{Rad}(A \otimes B) = \operatorname{Rad}(A) \otimes B + A \otimes \operatorname{Rad}(B)$. Indeed, denote the latter by J. Then J is a nilpotent ideal in $A \otimes B$, as $\operatorname{Rad}(A)$ and $\operatorname{Rad}(B)$ are nilpotent. On the other hand, $(A \otimes B)/J = (A/\operatorname{Rad}(A)) \otimes (B/\operatorname{Rad}(B))$, which is a product of two semisimple algebras, hence semisimple. This implies $J \supset \operatorname{Rad}(A \otimes B)$. Altogether, by Proposition 2.11, we see that $J = \operatorname{Rad}(A \otimes B)$, proving the claim.

Thus, we see that

$$(A \otimes B)/\operatorname{Rad}(A \otimes B) = A/\operatorname{Rad}(A) \otimes B/\operatorname{Rad}(B).$$

Now, M is an irreducible representation of $(A \otimes B)/\operatorname{Rad}(A \otimes B)$, so it is clearly of the form $M = V \otimes W$, where V is an irreducible representation of $A/\operatorname{Rad}(A)$ and W is an irreducible representation of $B/\operatorname{Rad}(B)$, and V, W are uniquely determined by M (as all of the algebras involved are direct sums of matrix algebras). $\square$