18.705 Commutative Algebra
Fall 2008

*** Problem SLK 1** (The '*' means that this problem is to be presented in class.) Let B be a ring, I an ideal, and $A := B[y]$ the polynomial ring. Construct an isomorphism from A/IA onto $(B/I)[y]$.

Problem SLK 2 Let B be a UFD, and $A := B[y]$ the polynomial ring. Let f be a polynomial that has a term by^i with $i > 0$ such that b is not divisible by some prime element p in B. Prove that the ideal (f) is not maximal.

Problem SLK 3 Let L, M, N be A-modules, and $\alpha\colon L \to M$, $\beta\colon M \to N$, $\sigma\colon N \to M$, $\rho\colon M \to L$ homomorphisms. Prove that $M = L \oplus N$ and $\alpha = i_L$, $\beta = \pi_N$, $\sigma = i_N$, $\rho = \pi_L$ if and only and if and only if the following relations hold: $\beta\alpha = 0$, $\beta\sigma = 1$, $\rho\sigma = 0$, $\rho\alpha = 1$, and $\alpha\rho + \sigma\beta = 1$.

Problem SLK 4 Let k be a field, and K an algebraically closed field containing k. (Recall that K contains a copy of every algebraic extension of k.) Let A be the polynomial ring in n variables over k, and $f, f_1, \ldots, f_r$ polynomials in A. Suppose that, for any n-tuple $a := (a_1, \ldots, a_n)$ of elements a_i of K such that $f_1(a) = 0, \ldots, f_r(a) = 0$, also $f(a) = 0$. Prove that there are an integer N and polynomials $g_1, \ldots, g_r$ in A such that $f^N = g_1 f_1 + \cdots + g_r f_r$.

Problem SLK 5 Let A be a ring, and P a module. Then P is called *projective* if the functor $N \mapsto \mathrm{Hom}(P, N)$ is exact. (1) Prove that P is projective if and only if, given any surjection $\psi\colon M \to N$, every map $\nu\colon P \to N$ lifts to a map $\mu\colon P \to M$; that is, $\psi\mu = \nu$. (2) Prove that P is projective if and only if every short exact sequence $0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} P \to 0$ is split. (3) Prove that P is projective if and only if P is a direct summand of a free module F; that is, $F = P \oplus L$ for some L. (4) Assume that A is local and that P is finitely generated; then prove that P is projective if and only if P is free.

Problem SLK 6 Let A be a Noetherian ring, and P a finitely generated A-module. Prove that the following three conditions are equivalent: (1) P is projective; (2) $P_{\mathfrak{p}}$ is free over $A_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$; and (3) $P_{\mathfrak{m}}$ is free over $A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$.

Problem SLK 7 Let A be a ring, M an arbitrary A-module, and I the annihilator of M. Prove that the support $\operatorname{Supp}(M)$ is always contained in the set $\mathbb{V}(I)$ of primes containing I.

Problem SLK 8 Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ the rational numbers, and set $M := \mathbb{Q}/\mathbb{Z}$. Find the support $\operatorname{Supp}(M)$, and show that it's not Zariski closed (that is, it does not consist of all the primes containing any ideal).

Problem SLK 9 Let A be a Noetherian ring, M a finitely generated module. Prove that the intersection of all the associated primes of M is equal to the radical of the annihilator $\operatorname{Ann}(M)$.

* **Problem SLK 10** Let A be a Noetherian ring, I and J ideals. Assume JA_P is contained in IA_P for all associated primes P of A/I. Prove J is contained in I.

* **Problem SLK 11** Let A be a Noetherian ring, $x \in A$. Assume x lies in no associated prime of A/I. Prove the intersection of the ideals (x) and I is equal to their product $(x)I$.

Problem SLK 12 Let A be a Noetherian ring, M a finitely generated module, Q a submodule. Set $P := \sqrt{\mathrm{Ann}(M/Q)}$. Prove the equivalence of these two conditions:

(1) Q is P-primary; that is, $\mathrm{Ass}(M/Q) = \{P\}$; and

(2) every zero divisor on M/Q is nilpotent on M/Q; in other words, given an $a \in A$ for which there exists an $x \in M - Q$ such that $ax \in Q$, necessarily $a \in P$.

Problem SLK 13 Let A be a domain, K its fraction field. Show that A is a valuation ring if and only if, given any two ideals I and J, either I lies in J or J lies in I.

* **Problem SLK 14** Let v be a valuation of a field K, and $x_1, \ldots, x_n$ nonzero elements of K with $n > 1$. Show that (1) if $v(x_1)$ and $v(x_2)$ are distinct, then $v(x_1 + x_2) = \min\{v(x_1), v(x_2)\}$ and that (2) if $x_1 + \cdots + x_n = 0$, then $v(x_i) = v(x_j)$ for two distinct indices i and j.

Problem SLK 15 Prove that a valuation ring is normal.

Problem SLK 16 Let A be a Dedekind domain. Suppose A is *semilocal* (that is, A has only finitely many maximal ideals). Prove A is a PID.

Problem SLK 17 Let A be a Noetherian ring, and suppose A_P is a domain for every prime P. Prove the following four statements:

(1) Every associated prime of A is minimal.
(2) The ring A is reduced.
(3) The minimal primes of A are pairwise coprime.
(4) The ring A is equal to the product of its quotients A/P as P ranges over the set of all minimal primes.

Problem SLK 18 Let A be a UFD, and M an invertible fractional ideal. Prove M is principal.

* **Problem SLK 19** Let A be a domain, K its fraction field, L a finite extension field, and B the integral closure of A in L. Show that L is the fraction field of B. Show that, in fact, every element of L can be expressed as a fraction b/a where b is in B and a is in A.

Problem SLK 20 Let $A \subset B$ be domains, and K, L their fraction fields. Assume that B is a finitely generated A-algebra, and that L is a finite dimensional K-vector space. Prove that there exists an $f \in A$ such that B_f is a finite generated A_f-module.

Problem SLK 21 Let A be a ring, P a prime ideal, and B an integral extension ring. Suppose B has just one prime Q over P. Show (a) that QB_P is the only maximal ideal of B_P, (b) that $B_Q = B_P$, and (c) that B_Q is integral over A_P.

Problem SLK 22 Let A be a ring, P a prime ideal, B an integral extension ring. Suppose B is a domain, and has at least two distinct primes Q and Q' over P. Show B_Q is not integral over A_P. Show, in fact, that if x lies in Q', but not in Q, then $1/x \in B_Q$ is not integral over A_P.

Problem SLK 23 Let k be a field, and x an indeterminate. Set $B := k[x]$, and set $y := x^2$ and $A := k[y]$. Set $P := (y-1)A$ and $Q := (x-1)B$. Is B_Q is integral over A_P? Explain.

* **Problem SLK 24** Let A be a ring (possibly not Noetherian), P a prime ideal, and B a module-finite A-algebra. Show that B has only finitely many primes Q over P. [Hint: reduce to the case that A is a field by localizing at P and passing to the residue rings.]

Problem SLK 25 Let k be a field, A a finitely generated k-algebra, and f a nonzero element of A. Assume A is a domain. Prove that A and its localization A_f have the same dimension.

Problem SLK 26 Let A be a DVR, and f a uniformizing parameter. Show that A and its localization A_f do NOT have the same dimension.

Problem SLK 27 Let L/K be an algebraic field extension. Let $X_1, \ldots, X_n$ be indeterminates, and A and B the corresponding polynomial rings over K and L. (1) Let Q be a prime of B, and P its contraction in A. Show $\mathrm{ht}(P) = \mathrm{ht}(Q)$. (2) Let f and g be two polynomials in A with no common factors in A. Show f and g have no common factors in B.

* **Problem SLK 28** Let k be a field, and A a finitely generated k-algebra. Prove that A is Artin if and only if A is a finite-dimensional k-vector space.

Problem SLK 29 Let A be an r-dimensional finitely generated domain over a field, and x an element that's neither 0 nor a unit. Set $B := A/(x)$. Prove that B is equidimensional of dimension $r-1$ (that is, $\dim(B/Q) = r-1$ for every minimal prime Q); prove that, in fact, $r-1$ is the length of any maximal chain of primes in B.

* **Problem SLK 30** Let $A, \mathbf{m}$ be a Noetherian local ring. Assume that $\mathbf{m}$ is generated by an A-sequence $x_1, \ldots, x_r$. Prove that A is regular of dimension r.

Problem SLK 31 Let $A, \mathbf{m}$ be a Noetherian local ring of dimension r, and $B := A/I$ a factor ring of dimension s. Set $t := r - s$. Prove that the following three conditions are equivalent: (1) A is regular, and I is generated by t members of a regular sop; (2) B is regular, and I is generated by t elements; and (3) A and B are regular.

Problem SLK 32 (a) Let A be a Noetherian local ring, and P a principal prime ideal of height 1. Prove that A is a domain.

(b) Let k be a field, and x an indeterminate. Show that the product ring $k[x] \times k[x]$ is not a domain, yet it contains a principal prime ideal P of height 1.

Problem SLK 33 (a) Let A be a ring, S a multiplicative set, and M an A-module. Prove that $S^{-1}M = S^{-1}A \otimes M$ by showing that the two natural maps $M \to S^{-1}M$ and $M \to S^{-1}A \otimes M$ enjoy the same universal property.

(b) Show that $(1, 1, \dots)$ is nonzero in $\mathbb{Q} \otimes \left(\prod_i \mathbb{Z}/(i)\right)$.

* **Problem SLK 34** Let A be a ring, I and J ideals, and M an A-module.

(a) Use the right exactness of tensor product to show that $(A/I) \otimes M = M/IM$.

(b) Show that $(A/I) \otimes (A/J) = A/(I + J)$.

(c) Assume that A is a local ring with residue field k, and that M is finitely generated. Show that $M = 0$ if and only if $M \otimes k = 0$.

(d) Let $\mathbb{R}$ be the real numbers, $\mathbb{C}$ the complex numbers, and X an indeterminate. Using the formula $\mathbb{C} = \mathbb{R}[X]/(1 + X^2)$, express $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ as a product of Artin local rings, identifying the factors.

Problem SLK 35: Let A be an arbitary ring, M and N A-modules, and k a field.

(a) Assume M and N are free of ranks m and n. Prove that $M \otimes N$ is free of rank mn.

(b) Given nonzero k-vector spaces V and W, show that $V \otimes W$ is also nonzero.

(c) Assume A is local, and M and N are finitely generated. Prove that $M \otimes N = 0$ if only only if $M = 0$ or $N = 0$.

(d) Assume M and N are finitely generated. Prove $\mathrm{Supp}(M \otimes N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$.