As in the previous lecture, let $\mathcal{H} = \{h : \mathcal{X} \mapsto [-1,1]\}$ be a VC-subgraph class and $f \in \mathcal{F} = \text{conv } \mathcal{H}$. The classifier is $\text{sign}(f(x))$. The set

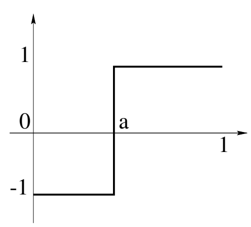
$$\{y \neq \text{sign}(f(x))\} = \{yf(x) \leq 0\}$$

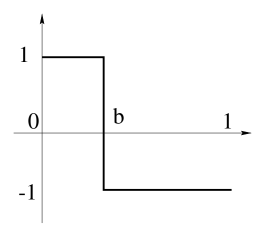
is the set of misclassified examples and $\mathbb{P}\left(yf(x) \leq 0\right)$ is the misclassification error.
Assume the examples are labeled according to $C_0 = \{x \in \mathcal{X} : y = 1\}$. Let $C = \{\text{sign}(f(x)) > 0\}$. Then $C_0 \triangle C$ are misclassified examples.

$$\mathbb{P}\left(C \triangle C_0\right) = \frac{1}{n}\sum_{i=1}^{n} I(x_i \in C\triangle C_0) + \underbrace{\mathbb{P}\left(C\triangle C_0\right) - \frac{1}{n}\sum_{i=1}^{n} I(x_i \in C\triangle C_0)}_{\text{small. estimate uniformly over sets } C} .$$

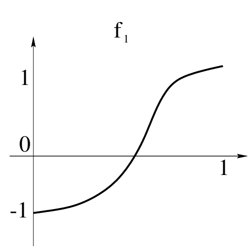
For voting classifiers, the collection of sets $\mathcal{C}$ can be "very large".

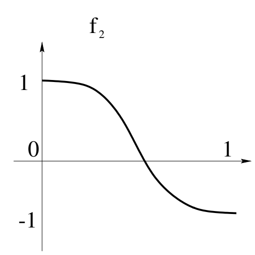
**Example 20.1.** Let $\mathcal{H}$ be the class of simple step-up and step-down functions on the $[0,1]$ interval, parametrized by $a$ and $b$.






Then $VC(\mathcal{H}) = 2$. Let $\mathcal{F} = \text{conv } \mathcal{H}$. First, rescale the functions: $f = \sum_{i=1}^{T} \lambda_i h_i = 2\sum_{i=1}^{T} \lambda_i \left(\frac{h_i+1}{2}\right) - 1 = 2f' - 1$ where $f' = \sum_{i=1}^{T} \lambda_i h_i'$, $h_i' = \frac{h_i+1}{2}$. We can generate any non-decreasing function $f'$ such that $f'(0) = 0$ and $f'(1) = 1$. Similarly, we can generate any non-increasing $f'$ such that $f'(0) = 1$ and $f'(1) = 0$. Rescaling back to $f$, we can get any non-increasing and non-decreasing functions of the form

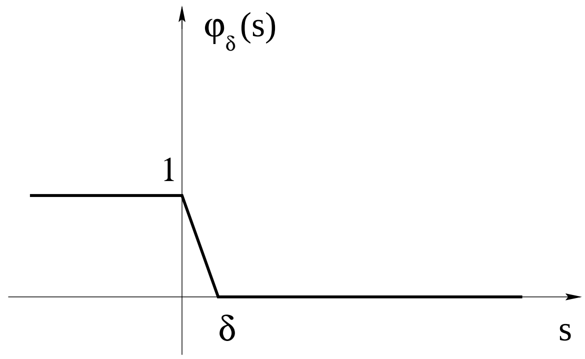





Any function with sum of jumps less than 1 can be written as $f = \frac{1}{2}(f_1 + f_2)$. Hence, we can generate basically all sets by $\{f(x) > 0\}$, i.e. conv $\mathcal{H}$ is bad.

Recall that $\mathbb{P}\left(yf(x) \leq 0\right) = \mathbb{E}I(yf(x) \leq 0)$. Define function $\varphi_\delta(s)$ as follows:
Then,

$$I(s \leq 0) \leq \varphi_\delta\left(s\right) \leq I(s \leq \delta).$$




Hence,

$$
\begin{aligned}
\mathbb{P}\left(yf(x)\le 0\right) &\le \mathbb{E}\varphi_\delta\left(yf(x)\right)\\
&= \frac{1}{n}\sum_{i=1}^{n}\varphi_\delta\left(y_i f(x_i)\right) + \left(\mathbb{E}\varphi_\delta\left(yf(x)\right) - \frac{1}{n}\sum_{i=1}^{n}\varphi_\delta\left(y_i f(x_i)\right)\right)\\
&\le \frac{1}{n}\sum_{i=1}^{n} I(y_i f(x_i)\le\delta) + \left(\mathbb{E}\varphi_\delta\left(yf(x)\right) - \frac{1}{n}\sum_{i=1}^{n}\varphi_\delta\left(y_i f(x_i)\right)\right)
\end{aligned}
$$

By going from $\frac{1}{n}\sum_{i=1}^{n} I(y_i f(x_i)\le 0)$ to $\frac{1}{n}\sum_{i=1}^{n} I(y_i f(x_i)\le \delta)$, we are penalizing small confidence predictions. The margin $yf(x)$ is a measure of the confidence of the prediction.
For the sake of simplicity, denote $\mathbb{E}\varphi_\delta = \mathbb{E}\varphi_\delta\left(yf(x)\right)$ and $\bar{\varphi_\delta} = \frac{1}{n}\sum_{i=1}^{n}\varphi_\delta\left(y_i f(x_i)\right)$.

**Lemma 20.2.** *Let $\mathcal{F}_d = conv_d\ \mathcal{H} = \{\sum_{i=1}^{d}\lambda_i h_i, h_i\in\mathcal{H}\}$ and fix $\delta\in(0,1]$. Then*

$$
\mathbb{P}\left(\forall f\in\mathcal{F}_d,\ \frac{\mathbb{E}\varphi_\delta - \bar{\varphi_\delta}}{\sqrt{\mathbb{E}\varphi_\delta}} \le K\left(\sqrt{\frac{dV\log\frac{n}{\delta}}{n}} + \sqrt{\frac{t}{n}}\right)\right) \ge 1 - e^{-t}.
$$

*Proof.* Denote

$$
\varphi_\delta\left(y\mathcal{F}_d(x)\right) = \{\varphi_\delta\left(yf(x)\right), f\in\mathcal{F}_d\}.
$$

Note that $\varphi_\delta\left(yf(x)\right) : \mathcal{X}\times\mathcal{Y}\mapsto[0,1]$.
For any $n$, take any possible points $(x_1,y_1),\ldots,(x_n,y_n)$. Since

$$
|\varphi_\delta\left(s\right) - \varphi_\delta\left(t\right)| \le \frac{1}{\delta}|s-t|,
$$

we have

$$\begin{aligned} d_{x,y}\left(\varphi_\delta\left(yf(x)\right), \varphi_\delta\left(yg(x)\right)\right) &= \left(\frac{1}{n}\sum_{i=1}^{n}\left(\varphi_\delta\left(y_i f(x_i)\right) - \varphi_\delta\left(y_i g(x_i)\right)\right)^2\right)^{1/2} \\ &\le \left(\frac{1}{\delta^2}\frac{1}{n}\sum_{i=1}^{n}\left(y_i f(x_i) - y_i g(x_i)\right)^2\right)^{1/2} \\ &= \frac{1}{\delta}\left(\frac{1}{n}\sum_{i=1}^{n}\left(f(x_i) - g(x_i)\right)^2\right)^{1/2} \\ &= \frac{1}{\delta} d_x(f,g) \end{aligned}$$

where $f, g \in \mathcal{F}_d$.

Choose $\varepsilon \cdot \delta$-packing of $\mathcal{F}_d$ so that

$$d_{x,y}\left(\varphi_\delta\left(yf(x)\right), \varphi_\delta\left(yg(x)\right)\right) \le \frac{1}{\delta} d_x(f,g) \le \varepsilon.$$

Hence,

$$\mathcal{N}(\varphi_\delta\left(y\mathcal{F}_d(x)\right), \varepsilon, d_{x,y}) \le \mathcal{D}(\mathcal{F}_d, \varepsilon\delta, d_x)$$

and

$$\log \mathcal{N}(\varphi_\delta\left(y\mathcal{F}_d(x)\right), \varepsilon, d_{x,y}) \le \log \mathcal{D}(\mathcal{F}_d, \varepsilon\delta, d_x) \le KdV \log \frac{2}{\varepsilon\delta}.$$

We get

$$\log \mathcal{D}(\varphi_\delta\left(y\mathcal{F}_d\right), \varepsilon/2, d_{x,y}) \le KdV \log \frac{2}{\varepsilon\delta}.$$

So, we can choose $f_1, \ldots, f_D$, $D = \mathcal{D}(\mathcal{F}_d, \varepsilon\delta, d_x)$ such that for any $f \in \mathcal{F}_d$ there exists $f_i$, $d_x(f, f_i) \le \varepsilon\delta$. Hence,

$$d_{x,y}(\varphi_\delta\left(yf(x)\right), \varphi_\delta\left(yf_i(x)\right)) \le \varepsilon$$

and $\varphi_\delta\left(yf_1(x)\right), \ldots, \varphi_\delta\left(yf_D(x)\right)$ is an $\varepsilon$-cover of $\varphi_\delta\left(y\mathcal{F}_d(x)\right)$. □