18.465 PS3, due Tuesday, March 8, 2005

1. Let $f_0(x) = 1 - |x|$ for $|x| \leq 1$ and 0 elsewhere. Let $f_\theta(x) = f_0(x - \theta)$. If we have i.i.d. observations $X_1, ..., X_n$ with density f_θ, we can estimate θ by (a) the sample mean $\overline{X}$, (b) the sample median m_n, or (c) the Hodges-Lehmann estimator $\hat{\theta}_{HL}$. For each estimator $T = T_n$, find the limiting distribution of $\sqrt{n}(T_n - \theta)$. A smaller variance means the estimator is more efficient. Rank the estimators in order of efficiency in this case.

2. For f_0 a Cauchy distribution, $\overline{X}$ no longer converges to θ, so one might say that the asymptotic variance is infinite. Proceed as in problem 1 for the other two estimators and compare them for efficiency.

3. Randles and Wolfe, problem 11.5.4, but omit the distribution-free question, just find the null distribution.

4. Let F_m be an empirical distribution function for F and independent of it let G_n be an empirical distribution function for G, each based on i.i.d. observations. Let

$$\zeta_{m,n}(x) \ := \ \sqrt{\frac{mn}{m+n}}(F_m - G_n)(x).$$

Under the null hypothesis $H_0: \ F = G$, $\zeta_{m,n}$ converges as $m, n \to \infty$ in distribution to $y_{F(x)}$ where y_t, $0 \leq t \leq 1$ is a Brownian bridge process. Let $KS_{m,n}$ be the Kolmogorov-Smirnov statistic

$$KS_{m,n} \ := \ \sup_x |\zeta_{m,n}(x)|.$$

If $F = G$ is continuous then for large m, n, this has asymptotically the distribution of $\sup_{0<t<1} |y_t|$ given by RAP, Proposition 12.3.4. Assuming that the asymptotic distribution is valid, would H_0 be rejected at the $\alpha = 0.05$ level if $KS_{m,n} = 1.5$?

5. Find an n_0 such that for $m, n \geq n_0$ it can be proved from the Bretagnolle-Massart theorem 1.1 that $P_0(KS_{m,n} \geq 2) \leq 0.05$. *Hint*: In the Bretagnolle-Massart handout, make the right side of (1.2) ≤ 0.005. This determines x. Find η such that $P(\sup_t |y_t| \geq \eta) \leq 0.04$ and find n large enough so that $(x + c \cdot \log n)/\sqrt{n} \leq (2 - \eta)/2$ Try $n_0 = v \cdot 10^5$ for some single-digit integer v.