18.443 Problem Set 1 Spring 2014
Statistics for Applications
Due Date: 2/13/2015
prior to 3:00pm

Problems from John A. Rice, Third Edition. [*Chapter.Section.Problem*]

1. Problem 6.4.1

   $Z \sim N(0,1)$ and $U \sim \chi^2_n$ and Z and U are independent.

   $T = Z/\sqrt{U/n}$ a Student's t random variable with n degrees of freedom.

   Find the density function of T.

   Solution:

   - The density of Z is $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2}$, $-\infty < x < +\infty$.
   - The density of U is
   $$f_U(u) = \begin{cases} \frac{1}{\Gamma(n/2)2^{n/2}} u^{(n/2)-1} e^{-u/2} & ,if \quad u > 0 \\ 0 & ,if \quad u \le 0 \end{cases}$$
   - Because Z and U are independent their joint density is
   $$f_{Z,U}(z,u) = f_Z(z) f_U(u)$$
   - Consider transforming (Z,U) to (T,V), where
   $T = Z/\sqrt{U/n}$ and $V = U$,
   computing the joint density of (T,V) and then integrating out V to obtain the marginal density of T.
     - Determine the functions $g(T,V) = Z$ and $h(T,V) = U$
     $$g(T,V) = \sqrt{V/n}T$$
     $$h(T,V) = V$$
     Then the joint density of (T,V) is given by
     $$f_{T,V}(t,v) = f_{Z,U}(g(t,v), h(t,v)) \times J$$
     where J is the Jacobian of the transformation from (Z,U) to (Z,U).
     Compute J:

$$J = \begin{vmatrix} \frac{\partial g(t,v)}{\partial t} & \frac{\partial g(t,v)}{\partial v} \\ \frac{\partial h(t,v)}{\partial t} & \frac{\partial h(t,v)}{\partial v} \end{vmatrix} = \begin{vmatrix} \sqrt{V/n} & (\frac{T}{\sqrt{n}})\frac{1}{2}V^{-1/2} \\ 0 & 1 \end{vmatrix} = \sqrt{V/n}$$

The joint density of (T, V) is thus

$$
\begin{aligned}
f_{T,V}(t,v) &= f_Z(g(t,v)) \times f_U(h(t,v)) \times J \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{v/n}t)^2} \times \frac{1}{\Gamma(n/2)2^{n/2}} v^{(n/2)-1} e^{-v/2} \times \sqrt{v/n} \\
&= \frac{1}{\sqrt{2\pi n}\Gamma(n/2)2^{n/2}} v^{\frac{1}{2}(n+1)-1} e^{-\frac{v}{2}(1+\frac{t^2}{n})}
\end{aligned}
$$

Integrate over v to obtain the marginal density of T:

$$
\begin{aligned}
f_T(t) &= \int_0^\infty f_{T,V}(t,v)dv \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi n}\Gamma(n/2)2^{n/2}} v^{\frac{1}{2}(n+1)-1} e^{-\frac{v}{2}(1+\frac{t^2}{n})} dv \\
&= \frac{1}{\sqrt{2\pi n}\Gamma(n/2)2^{n/2}} \int_0^\infty v^{\frac{1}{2}(n+1)-1} e^{-\frac{v}{2}(1+\frac{t^2}{n})} dv
\end{aligned}
$$

The integral factor can be evaluated by recognizing that it is identical to integrating a $Gamma(\alpha, \lambda)$ density function apart from the normalization constant, with $\alpha = (n+1)/2$ and $\lambda = \frac{1}{2}(1 + \frac{t^2}{n})$, that is

$$
\begin{aligned}
1 &= \int_0^\infty \tfrac{1}{\Gamma(\alpha)} \lambda^\alpha v^{\alpha-1} e^{-\lambda v} dv \\
\text{So } \Gamma(\alpha)\lambda^{-\alpha} &= \int_0^\infty v^{\alpha-1} e^{-\lambda v} dv
\end{aligned}
$$

which gives

$$
\begin{aligned}
\int_0^\infty v^{\frac{1}{2}(n+1)-1} e^{-\frac{v}{2}(1+\frac{t^2}{n})} dv &= \Gamma(\alpha) \times \lambda^{-\alpha} \\
&= \Gamma((n+1)/2) \times [\tfrac{1}{2}(1 + \tfrac{t^2}{n})]^{-(n+1)/2}
\end{aligned}
$$

Finally we can write

$$
\begin{aligned}
f_T(t) &= \int_0^\infty f_{T,V}(t,v)dv \\
&= \frac{1}{\sqrt{2\pi n}\Gamma(n/2)2^{n/2}} \int_0^\infty v^{\frac{1}{2}(n+1)-1} e^{-\frac{v}{2}(1+\frac{t^2}{n})} dv \\
&= \frac{1}{\sqrt{2\pi n}\Gamma(n/2)2^{n/2}} \Gamma((n+1)/2) \times [\frac{1}{2}(1 + \frac{t^2}{n})]^{-(n+1)/2} \\
&= \frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\Gamma(n/2)} \times \left[1 + \frac{t^2}{n}\right]^{-(n+1)/2}
\end{aligned}
$$

2. Suppose the random variable X has a t distribution with n degrees of freedom.

   (a). For what values of n is the variance finite/infinite.

   (b). Derive a formula for the variance of X (when it is finite).

   Solution:

   (a). For the variance of the t distribution to be finite it must have a finite second moment:

$E[T^2] = \int t^2 f_T(t)dt < \infty.$

The integrand of this second moment calculation is proportional to

$$t^2 f_T(t) \propto \frac{t^2}{\left[1 + \frac{t^2}{n}\right]^{(n+1)/2}} \xrightarrow{t \to \infty} n^{(n+1)/2} \times t^{2-(n+1)} \propto t^{1-n}$$

The integral of this integrand thus converges if and only if

$(1 - n) < (-1)$, which is equivalent to $n > 2$.

(b). If $n > 2$, then the variance of T is finite. For such n, the mean of T exists and is zero, so writing $T = Z/\sqrt{U/n}$ for independent $Z \sim N(0, 1)$ and $U \sim \chi^2_n$

$$\begin{aligned} Var(T) = E[T^2] &= E[[Z/\sqrt{U/n}]^2] = E[nZ^2/U] \\ &= n \times E[Z^2] \times E\left[\tfrac{1}{U}\right] \\ &= 1 \times n \times \tfrac{1}{(n-2)} \\ &= \frac{n}{n-2} \end{aligned}$$

The expectation $E[\frac{1}{U}] = 1/(n-2)$ can be computed directly for $n > 2$.

(Note that the formula is undefined for $n = 2$ and gives negative values for $n < 2$)

3. 6.4.4. Also, add part (c) answer the question if the random variable T follows a standard normal distribution $N(0, 1)$. Comment on the differences and why that should be.

Solution:

We are given that T follows a t_7 distribution. The problem is solved by finding an expression for t_0 in terms of the cumulative distribution function of T.

(a). To find the t_0 such that $P(|T| < t_0) = .9$ this is equivalent to $P(T < .95)$, which is solved in R using the function $qt()$ – the quantile function for the t distribution

```
> args(qt)
function (p, df, ncp, lower.tail = TRUE, log.p = FALSE)

>qt(.95,df=7)
[1] 1.894579
```

So, $t_0 = 1.894579$.

(b). $P(T > t_0) = .05$ is equivalent to $P(T \leq t_0) = 1 - .05 = .95$. This is the same t_0 found in (a).

```
> args(qt)
function (p, df, ncp, lower.tail = TRUE, log.p = FALSE)

> qt(p=.95, df=7)
[1] 1.894579
# Which is equivalent to
> qt(p=.05, df=7, lower.tail=FALSE)
[1] 1.894579
```

(c). For the standard normal distribution we use $qnorm()$ – the quantile function for the $Normal(0, 1)$ distribution

```
> args(qnorm)
function (p, mean = 0, sd = 1, lower.tail = TRUE, log.p = FALSE)
NULL
> qnorm(.95)
[1] 1.644854

> qnorm(p=.05, lower.tail=FALSE)
[1] 1.644854
```

So for both parts (a) and (b) $t_0 = 1.644854$ for the $N(0, 1)$ r.v. versus $t_0 = 1.894579$ for the t distribution with 7 degrees of freedom.

The t_0 values are larger for the t distribution indicating that the t distribution has heavier tail areas than the $Normal(0, 1)$ distribution. This makes sense because the t distribution equals a $Normal(0, 1)$ random variable divided by a random variable with expectation equal to 1 but positive variance. The possibility of the denominator of the t ratio being less than 1 increases the probability of larger values.

4. Problem 8.10.10.

   Use the normal approximation of the Poisson distribution to sketch the approximate sampling distribution of $\hat{\lambda}$ of Example A of Section 8.4. According to this approximation, what is

$P(|\lambda_0 - \hat{\lambda}| > \delta)$ for $\delta = -.5, 1, 1.5, 2, 2.5$

where λ_0 is the true value of λ.

Solution:

In the example, the estimate $\hat{\lambda} = \overline{X} = \frac{1}{n}\sum_1^n X_i = 24.9$, with $n = 23$.

The $X_1, \ldots, X_n$ are assumed to be i.i.d. (independent and identically distributed) $Poisson(\lambda_0)$ random variables with

$$E[X_i] = \lambda_0 \text{ and } Var[X_i] = \lambda_0$$

By the Central Limit Theorem

$$\sqrt{n}\frac{(\overline{X}-\lambda_0)}{\sqrt{\lambda_0}} \xrightarrow{n\to\infty} N(0,1).$$

The approximate sampling distribution of $\hat{\lambda}$ is thus a Normal distribution centered λ_0 with

standard deviation equal to $\sqrt{\lambda_0/n} \approx \sqrt{\frac{\hat{\lambda}}{n}} = \sqrt{24.9/23} = 1.040485$.

For the probability computations:

$$\begin{aligned} P(|\lambda_0 - \hat{\lambda}| > \delta) &= P(|\lambda_0 - \overline{X}| > \delta) \\ &= P(\frac{\sqrt{n}|\lambda_0 - \overline{X}|}{\sqrt{\lambda_0}} > \sqrt{n}\frac{\delta}{\sqrt{\lambda_0}}) \\ &\approx P(|N(0,1)| > \sqrt{n}\frac{\delta}{\sqrt{\hat{\lambda}}}) \\ &= P(|N(0,1)| > \frac{\sqrt{23}}{\sqrt{24.9}}\delta) \\ &= 2 \times (1 - \Phi(\sqrt{23/24.9} \times \delta)) \end{aligned}$$

Using R and the function *pnorm* we can compute the desired values:

```
> 2 * 1-pnorm( sqrt(23/24.9) * c(.5,1.,1.5,2.,2.5))
[1] 1.315420 1.168253 1.074703 1.027292 1.008137
```

5. Problem 8.10.13.

In Example D of Secton 8.4, the method of moments estimate was found to be $\hat{\alpha} = 3\overline{X}$. In this example, consider the sampling distribution of $\hat{\alpha}$.

(a). Show that $E(\hat{\alpha}) = \alpha$, that is, that the estimate is unbiased.

(b). Show that $Var(\hat{\alpha}) = (3 - \alpha^2)/n$.

(c). Use the central limit theorem to deduce a normal approximation to the sampling distribution of $\hat{\alpha}$.

According to this approximation, if $n = 25$ and $\alpha = 0$, what is the $P(|\hat{\alpha}| > .5)$.

Solution

The sample of values $X_1, \ldots, X_n$ giving $\overline{X}$ are i.i.d. with density function

$$f(x \mid \alpha) = \tfrac{1+\alpha x}{2}, \text{ for } -1 \le x \le +1,$$

with parameter $\alpha : -1 \le \alpha \le 1$. (The values are such that $x_i = cos(\theta_i)$, where θ_i is the angle at which electrons are emitted in muon decay.)

(a). Since the X_i are i.i.d.

$$E[\overline{X}] \;=\; E[X_i] = \textstyle\int_{-1}^{1} x \times (\frac{1+\alpha x}{2}) dx = \alpha/3$$

It follows that

$$E[\hat{\alpha}] = E[3\overline{X}] = 3E[\overline{X}] = 3(\alpha/3) = \alpha$$

(b). Since the X_i are i.i.d.

$$\begin{aligned} Var[\overline{X}] &= Var[X_i]/n = (\tfrac{1}{n}) \times (E[X^2] - E[X]^2) \\ &= (\tfrac{1}{n}) \times ([\textstyle\int_{-1}^{1} x^2 \times (\frac{1+\alpha x}{2}) dx] - (\alpha/3)^2) \\ &= (\tfrac{1}{n}) \times ([\textstyle\int_{-1}^{1} \frac{x^2}{2} dx] - (\alpha/3)^2) \\ &= (\tfrac{1}{n}) \times ([1/3] - (\alpha/3)^2) \\ &= \frac{3 - \alpha^2}{9n} \end{aligned}$$

It follows that:

$$Var[\hat{\alpha}] = Var[3\overline{X}] = 9 \times Var[\overline{X}] = \frac{3 - \alpha^2}{n}.$$

(c) By the central limit theorm, for true parameter $\alpha = \alpha_0$, it follows that

$$\hat{\alpha} \xrightarrow{n \to \infty} N(\alpha_0, \frac{3 - \alpha_0^2}{n})$$

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