18.433 Combinatorial Optimization

The Matching Polytope: General graphs

September 23 *Lecturer: Santosh Vempala*

A matching M corresponds to a vector $x^M = (0,0,1,1,0...0)$ where x_e^M is 1 iff $e \in M$ and 0 if $e \notin M$. Let $\mathcal{M}$ be the convex hull of all vectors corresponding to matchings.

$$\mathcal{M} = conv\{x = \chi^M \mid M \text{ is a matching}\}$$

and the resulting relaxation of the integral constraints:

$$P = \{x \mid \quad x_e \geq 0 \quad \forall e \ \in E, \sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V\}$$

In the last lecture we saw that $\mathcal{M}(G) = P$ for any bipartite graph G. Also to describe the perfect matching polytope, $\mathcal{PM}(G)$, we just modified P by replacing the inequalities at each vertex by equalities.

What about for non-bipartite graphs? Clearly the same constraints do not apply, because we can have a triangle with edge weights of $\frac{1}{2}$ on each side. In such a situation, $P(G) \neq M(G)$ because there exist no perfect matchings of G, while the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in P(G)$.

In addition to the already mentioned constraints,

Constraint 1: $x_e \geq 0 \quad \forall e \in E$
Constraint 2: $\sum_{e \epsilon \delta(v)} x_e = 1 \quad \forall v \in V$

we require an additional constraint in the case of general graphs. This constraint is described below.

Constraint 3': Given a subgraph, $S \subseteq V$, of G such that $|S|$ is odd, then the number of edges within S that we can add to the matching is limited to:

$$\sum_{e \in S} x_e \leq \frac{|S| - 1}{2}$$

Now, there is a simpler way to state this third constraint set, which takes into account the fact that, for a perfect matching to exist, then G must have an even number of vertices. If we assume that $|V|$ is even, then by dividing G into S and $\overline{S}$, we obtain:

$$\sum_{e \in S} x_e \leq \frac{|S|-1}{2}$$
$$\sum_{e \in \overline{S}} x_e \leq \frac{|\overline{S}|-1}{2}$$

Here we note that both S and $\overline{S}$ have an odd number of vertices.

$$\Rightarrow \sum_{e \in S, e \in \overline{S}} x_e \leq \frac{|S|+|\overline{S}|}{2} - 1$$

Since we are considering only perfect matchings, it follows that

$$\sum_{e \in E} x_e = \frac{|V|}{2}$$

hence we obtain the following inequality, which for perfect matchings is equivalent to constraint 3'.

Constraint 3: $\sum_{e \in (S,\overline{S})} x_e \geq 1$ for any S of size odd.

This next theorem of Edmonds states that these three conditions determine the perfect matching polytope of any graph.

Theorem 1. *(Edmonds)*

$$P = \{x \mid x \ \ \textit{satisfies Constraints 1, 2, and 3}\} \ = \ PM(G)$$

Proof. It is fairly easy to show that $PM(G) \subseteq P$ because the constraints of P are satisfied by any perfect matching.

We will show that $P \subseteq PM(G)$ through contradiction.

Suppose that $P \not\subseteq PM(G)$. Let G be the smallest counterexample (by smallest, we mean the fewest number of edges). The fact that this is a counterexample implies that P has an x such that $x \notin PM(G)$. We will make a series of deductions from the assumptions that i) G is the smallest counterexample, and ii) $x \notin PM(G)$.

1. $0 < x_e < 1 \quad \forall\, e \in E$

   Suppose that for some e, $x_e = 0$, then we can delete the edge and obtain a smaller counterexample, $G - e$. But this is impossible, because we assumed G was the smallest counterexample. Also, if for some e, $x_e = 1$, then it must be disconnected from the graph (each vertex incident to the edge has no other neighbors by our second constraint condition). But if that is the case, then we can delete the endpoints of e and still have a counterexample, which again is a contradiction.

2. G has no isolated vertices (see Constraint 3).

3. G has no vertices of degree 1.

   If there was a vertex of degree 1, then the edge attached to that vertex would have weight 1, and we already showed in the first observation that it was impossible for $x_e = 1$. Hence each vertex has degree at least two.

4. There exists at least one vertex of degree strictly greater than 2.

   Suppose otherwise. If every vertex has degree 2, then we have some disjoint collection of cycles. For such a graph, it is easy to show that $P = PM(G)$.

5. $|E| > |V|$.

   This follows because:

$$2|E| = \sum_v \deg(v) > 2|V| \Rightarrow |E| > |V|$$

Note that P and $PM(G)$ are in $m = |E|$-dimensional space. So a vertex of P has to be the unique solution of for m independent constraints. So the vertex x must be the solution of m equalities. These m equalities must come from our base set of constraints:

$$\begin{array}{rcll} x_e & \geq & 0 & \forall\, e \in E \\ \sum_{e \in \delta(v)} x_e & = & 1 & \forall\, v \in V \\ \sum_{e \in \delta(S)} x_e & \geq & 1 & \forall\, S \subseteq V,\ |S| \text{ is odd} \end{array}$$

Looking at the three constraints, we see that we get 0 equalities from the first constraint, and $n = |V|$ from the second constraint. Since $m > n$, we must therefore get at least one constraint of the third type.

$$\Rightarrow \exists\, W, \quad \sum_{e \in \delta(W)} x_e = 1, \quad |W| \text{ odd and } \geq 3$$

Look at the cut $(W, \overline{W})$. The sum of the edges weights of the cut is 1. Form a new graph by contracting $\overline{W}$ to a single vertex, u. Call the resulting graph G'. The edge variables are then redefined as follows

$$x'_e = \begin{cases} x_e & \text{if } e \in E(W) \\ x_{wu} = \sum_{v \in \overline{W}} x_{wv} & \text{if } wv \in (W, \overline{W}) \end{cases}$$

That is, x_e stays the same for edges not crossing the cut, and all x_e's are summed together for edges crossing the cut and coming from the same point in W.

We can easily check that the vector x' satisfies the three constraint conditions for G'. So if M' refer to matchings in G', with incidence vectors $\chi^{M'}$ then

$$x' \in PM(G')$$
$$\Rightarrow x' = \sum_{M'} \lambda_{M'} \chi^{M'}$$

This is a convex combination, so $\lambda_{M'} \geq 0$ and $\sum_{M'} \lambda_{M'} = 1$.

If we follow the same procedure as above, except contracting W rather than $\overline{W}$ to get G'', then we will get an $x'' \in PM(G'')$. It follows that

$$x'' = \sum_{M''} \alpha_{M''} \chi^{M''}$$

Using these decompositions of x' and x'', we can give an intuitive description of how to construct a decomposition of x into a convex composition of perfect matchings of the original graph (thus contradicting our initial assumption concerning G).

The basic idea is to use the fact Kx' and Kx'', for some large interger K, can be viewed as the sum of K incidence vectors from their respective matchings, M' and M''. Then, by finding the corresponding matchings in G, we can combine them to give perfect matchings in G. We can then use this to show that x is a convex combination of perfect matching vectors in G.

First we have that

$$\Rightarrow x' = \sum_{M'} \lambda_{M'} \chi^{M'} \tag{1}$$
$$\Rightarrow x'' = \sum_{M''} \alpha_{M''} \chi^{M''} \tag{2}$$

We will show that (1) and (2) together imply that x is a convex combination of perfect matchings.

For every perfect matching of M, we can associate perfect matchings M' and M'' in G' and G'' induced by M. We will use their coefficients in x' and x'' to define the coefficient of M.

We will show that x is a convex combination by simply finding one and showing that it is indeed a convex combination. Let

$$M = M' \cup M'' \text{ have weight } = \left(\frac{\lambda_{M'}\alpha_{M''}}{x_e}\right)$$

With this set of convex multipliers, x will be a convex combination. Take $e \in M' \cap M''$ then

$$x_e = \sum_{M' \colon e \in M'} \lambda_{M'} = \sum_{M'' \colon e \in M''} \alpha_{M''}$$

The following claim will give the result.

Claim 2.

$$x = \sum_{e \in (W, \overline{W})} \sum_{M \colon e \in M} \left(\frac{\lambda_{M'}\alpha_{M''}}{x_e}\right) \chi^M$$

Proof of Claim. Consider an edge f. Assume first that $f \in E(W)$. Then

$$\begin{aligned}
x_f &= \sum_{e \in (W, \overline{W})} \frac{1}{x_e} \sum_{M \colon e \in M} \lambda_{M'}\alpha_{M''}\chi_f^M \\
&= \sum_{e \in \delta(W)} \frac{1}{x_e} \sum_{M \colon e, f \in M} \lambda_{M'} \left(\sum_{M'' \colon e \in M''} \alpha_{M''} \right) \\
&= \sum_{e \in \delta(W)} \sum_{M' \colon e, f \in M'} \lambda_{M'} \\
&= \sum_{M' \colon f \in M'} \lambda_{M'} \\
&= x_f
\end{aligned}$$

Similarly arguments may be applied if $f \in E(\overline{W})$ or if $f \in (W, \overline{W})$. □

We have now shown x is a convex combination of perfect matchings, which means that it itself must be a perfect matching. But this contradicts our earlier assumption that $x \notin PM(G)$. Hence, we arrive at a contradiction from our original assumption that $P \not\subseteq PM(G)$, which means that $P \subseteq PM(G)$ proving the theorem. □