18.366: Random Walks and Diffusion
Lectures 6-7: Asymptotics of Fat Tails

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1 Introduction

A probability density function $p(x)$ is said to have a *power-law tail* if

$$p(x) \sim \frac{A}{|x|^{1+\alpha}} \ , \quad |x| \to \infty \tag{1}$$

for some real, positive constants A and α. We call A the *power-law tail amplitude* and α the *power-law tail exponent*. Note that in general A and α might be different for $x \to +\infty$ and $x \to -\infty$; however, we will only be considering symmetric density functions.

Clearly α must be positive in order for

$$\int_{-\infty}^{+\infty} p(x)\,dx = 1 \ .$$

Similarly, the n^{th} moment exists if and only if $n < \alpha$.

The nature of a PDF's power-law tails is strongly connected to the form of the PDF's characteristic function:

Conjecture (M. Bazant): Suppose $p(x)$ is a symmetric, continuous probability density function. Then $p(x)$ has a power-law tail with non-even-integral exponent α and amplitude A if and only if its cumulant generating function may be written

$$\psi(k) = f(k) + g(k) \tag{2}$$

where f is analytic at 0 and has the Taylor series

$$f(k) = \sum_{n=1}^{\infty} \frac{c_{2n}(ik)^{2n}}{(2n)!} \ , \tag{3}$$

and g is singular at 0 and has the asymptotic representation

$$g(k) \sim c_\alpha |k|^\alpha \ , \quad k \to 0 \tag{4}$$

where

$$c_\alpha = -\frac{A\pi}{\Gamma(\alpha+1)\sin(\alpha\pi/2)} \ . \tag{5}$$

1.1 Consequences

We define the coefficients c_{2n} of the Taylor series expansion for f to be *generalized cumulants*. For $2n < \alpha$, we know that c_{2n} is the order-$2n$ cumulant of the PDF. For $2n > \alpha$, however, the coefficients may exist despite the fact that the corresponding cumulants of the PDF don't exist. Similarly, we call A a *diverging cumulant*.

Suppose $p(x)$ is the PDF for each of the IID steps of a random walk, and let $P_N(x)$ be the PDF for the position of the walker after N steps. Even if the Conjecture is false, the cumulant generating function $\psi_N(k)$ for $P_N(x)$ must equal $N\psi(k)$. In particular, $\psi_N(k)$ has an analytic part of the form $Nf(k)$ and a singular part of the form $Ng(k)$. Regardless of their interpretation, then, the generalized cumulants and the diverging cumulants are additive (thus justifying the name *cumulants*). Since the Conjecture is an "if-and-only-if" statement, it would imply that $P_N(x)$ also has power-law tails with the same exponent as $p(x)$ and with amplitude NA. Equivalently, $P_N(x) \sim Np(x)$ as $|x| \to \infty$.

2 Motivation

We do not have a complete proof of the Conjecture (though it might be a known result in the field of Tauberian theorems). We do, however, have evidence in its favor.

2.1 Heuristics

In a random walk whose step lengths are bounded by some constant l, arriving at a position $x = O(Nl)$ after N steps can only be achieved by walking nearly the maximum distance to the right on nearly every step. This also holds (at least with probability exponentially approaching 1) for walks whose steps aren't bounded but for which the probability of large steps is exponentially small.

In a random walk whose step distribution has power-law tails, however, this isn't quite true. During computer simulations, such walks typically take many small steps interspersed with infrequent but very large steps. If a walker ends up far from the origin at time N, it is much more likely that it took a single large step than many small but coordinated steps. The probability that the walker is some very large distance from the origin at step N should therefore approximately equal the probability that one of its N steps was similarly large. We consequently expect $P_N(x) \sim Np(x)$ as $|x| \to \infty$.

2.2 Examples

2.2.1 Cauchy Distribution

The Cauchy distribution is defined by

$$p(x) = \frac{1}{\pi}\frac{1}{1+x^2} \ .$$

It clearly has power-law tails with amplitude $A = 1/\pi$ and exponent $\alpha = 1$. Because $\alpha = 1$, this distribution has no well-defined moments (not even a mean).

The characteristic function is

$$\hat{p}(k) = e^{-|k|} ,$$

so the cumulant generating function is

$$\psi(k) = -|k| .$$

This clearly satisfies the Conjecture.

2.2.2 Inverse-Quartic Distributions

The distribution defined by

$$p(x) = \frac{\sqrt{2}}{\pi} \frac{1}{1 + x^4}$$

has power-law tails with amplitude $A = \sqrt{2}/\pi$ and exponent $\alpha = 3$. The characteristic function is

$$\hat{p}(k) = e^{-|k|/\sqrt{2}} \left[\cos(k/\sqrt{2}) + \sin(|k|/\sqrt{2}) \right] ,$$

so the cumulant generating function is equal to its power series

$$\psi(k) = -\frac{1}{2}k^2 + \frac{\sqrt{2}}{6}|k|^3 - \frac{1}{6}k^4 + \cdots .$$

Similarly, the Student's t-distribution defined by

$$p(x) = \frac{2/\pi}{(1 + x^2)^2}$$

has characteristic function

$$\hat{p}(k) = e^{-|k|}(1 + |k|) .$$

Its cumulant generating function is equal to its power series

$$\psi(k) = -\frac{1}{2}k^2 + \frac{2}{6}|k|^3 - \frac{1}{4}k^4 + \cdots .$$

The Conjecture is satisfied in both cases.

3 Asymptotic Analysis

In the examples above, we started with PDFs that have power-law tails, and we showed that their cumulant generating functions must be of the form given by the Conjecture. In this section, we assume that the cumulant generating function for $p(x)$ is of the Conjectured form with $\alpha > 2$, and then find that for large N, $P_N(x)$ has the expected power-law tails.

Since $\hat{p}(k)$ and $P_N(x)$ are both even functions,

$$\begin{aligned} P_N(x) &= \int_{-\infty}^{+\infty} e^{ikx}\hat{p}(k)^N \,\frac{dk}{2\pi} \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos(kx)\hat{p}(k)^N \,dk \\ &= \frac{1}{\pi}\int_{0}^{+\infty} \cos(kx)\hat{p}(k)^N \,dk \ . \end{aligned} \tag{6}$$

The above expression is exact. Since $|\hat{p}(k)| \leq 1$ for all real k and since $\hat{p}(k=0) = 1$, we intuitively expect the integral 6 to be dominated by its behavior near $k = 0$ and for Laplace's method to be directly applicable. However, two problems can arise in getting global asymptotic information from the above: First, a generic characteristic function may approach or even reach $|\hat{p}(k)| = 1$ for some values of $k \neq 0$. Second, we have to make sure that every error made during Laplace's approximations are small even when x is large relative to N.

With enough care, we can apply Laplace's method as long as we "restrict" ourselves to values of x which grow at most polynomially in N: we shrink the upper limit of integration to something near 0; substitute in the asymptotic expansion for $\psi(k)$ near $k = 0$; keep the $\cos(kx)$ and $\exp(-N\sigma^2k^2/2)$ terms and Taylor-expand the other exponentials; and then re-raise the upper limit of integration to ∞. This yields (see Appendix A for tedious details):

$$\begin{aligned} P_N(x) \sim \frac{1}{\pi}&\int_0^{\infty} \cos(kx)\exp(-N\sigma^2k^2/2)\left[1 + (\text{higher } k^{2n} \text{ terms})\right]\,dk \\ &+ \frac{1}{\pi}\int_0^{\infty} \cos(kx)\exp(-N\sigma^2k^2/2)\left[Nc_\alpha k^\alpha\right]\,dk \ . \end{aligned} \tag{7}$$

For clarity and convenience, let $\phi(z)$ be the Gaussian density function with mean 0 and variance 1, and define $\phi_N(z) \equiv \sigma\sqrt{N}P_N(z\sigma\sqrt{N})$ to be the "normalized version" of P_N. Define the new variables $w \equiv \sigma\sqrt{N}k$, $z \equiv x/\sigma\sqrt{N}$, and $\lambda_\alpha \equiv c_\alpha/\sigma^\alpha$. Then equation 7 becomes

$$\begin{aligned} \phi_N(z) \sim \frac{1}{\pi}&\int_0^{\infty} \cos(wz)e^{-w^2/2}\left[1 + (\text{higher } w^{2n} \text{ terms})\right]\,dw \\ &+ \frac{\lambda_\alpha}{\pi N^{\alpha/2-1}}\int_0^{\infty} \cos(wz)e^{-w^2/2}w^\alpha\,dw \ . \end{aligned} \tag{8}$$

Now define

$$F_\beta(z) \equiv \frac{1}{\pi}\int_0^{\infty} \cos(wz)e^{-w^2/2}w^\beta\,dw \ . \tag{9}$$

Every term in equation 8 is of this form: the first integral is a sum of multiples of $F_{2n}(z)$ for integral n, and the second integral is a multiple of $F_\alpha(z)$. These integrals are evaluated in closed form in Appendix B. Substituting in the known values of $F_{2n}(z)$, equation 8 becomes

$$\phi_N(z) \sim \phi(z)\left[1 + (\text{higher-order Hermite terms})\right] + \frac{\lambda_\alpha}{N^{\alpha/2-1}}F_\alpha(z) \ . \tag{10}$$

Thus we see that $\phi_N(z)$ has the usual form described by the Central Limit Theorem with correction terms, but now there is an additional correction proportional to $F_\alpha(z)$. This additional correction term decreases algebraically, whereas $\phi(z)$ decreases exponentially. This has important consequences for the asymptotic behavior of $\phi_N(z)$.

First, we have seen that when the characteristic function is analytic, the width of the central region is not $\Delta x = O(\sqrt{N})$, but rather scales like some higher power of N. In this case, however, the $F_\alpha(z)$ correction is dominated by the first-order term if and only if

$$|z|^{-(\alpha+1)} N^{-(\alpha/2-1)} \ll e^{-z^2/2} .$$

Taking logarithms, this requirement becomes

$$-2(\alpha+1)\log|z| + z^2 \ll (\alpha-2)\log N .$$

Finally, noting that $\log|z| \ll z^2$ for large $|z|$, we require

$$z^2 \ll (\alpha-2)\log N .$$

Therefore the central region is now of width

$$\boxed{\Delta x = O\left(\sqrt{N\log N}\right)} .$$

It is only slightly wider than nominal ($\Delta x = O(\sqrt{N})$), and is much smaller than it would be if the cumulant generating function had been analytic.

Another consequence of the slow decay of $F_\alpha(z)$ is that this term provides the dominant behavior in the tails. Converting back to the original variables, this means that

$$P_N(x) \sim -\frac{c_\alpha \Gamma(\alpha+1)\sin(\alpha\pi/2)}{\pi}\frac{N}{|x|^{1+\alpha}} \ , |x| \to \infty.$$

Since we are assuming that c_α is of the Conjectured form (equation 5), then in fact

$$\boxed{P_N(x) \sim \frac{NA}{|x|^{1+\alpha}} \ , |x| \to \infty} . \tag{11}$$

That is, if the step distribution's characteristic function is of the Conjectured form, then $P_N(x)$ must have the corresponding fat tails.

Finally, see Chris Rycroft's handout for an example of how crucial the $F_\alpha(z)$ correction term can be even for small N and x.

A Why the Asymptotics are (Nearly) Global

We still need to justify the move from equation 6 to equation 7. To do this, we follow the standard steps for Laplace's method: shrink the interval of integration, substitute in the small-k asymptotic expansions of the integrand and throw out high-order terms, then re-enlarge the interval. During each approximation step, we must make sure that only sub-dominant errors are being made, and we must determine which values of x allow for such sub-dominance.

The first step (reducing the integration interval) requires a bit more knowledge of the behavior of $\hat{p}(k)$. Since it is the Fourier transform of a probability density function, we know that $|\hat{p}(k)| \leq 1$ for all real k. Since we're assuming that $p(x)$ is continuous, the Riemann-Lebesgue lemma guarantees that $|\hat{p}(k)| \to 0$ as $|k| \to \infty$. Therefore $|\hat{p}(k)|$ is bounded away from 1 for sufficiently large k, and by a relatively simple theorem about characteristic functions (Theorem 4.1.2 from [2]), $|\hat{p}(k)|$ must therefore be bounded away from 1 everywhere away from $k = 0$. More precisely, for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\hat{p}(k)| < (1 - \delta)\ \forall |k| > \epsilon$.

Given this information, it is clear that the error made in approximating 6 by

$$\frac{1}{\pi}\int_0^{\epsilon} \cos(kx)\hat{p}(k)^N \, dk \tag{12}$$

decreases with N like $(1 - \delta)^N$, *regardless of the magnitude of x.*

Now that we are restricted to a small neighborhood of $k = 0$, we can substitute in the asymptotic expression for $\psi(k)$:

$$\begin{aligned}
P_N(x) &\sim \frac{1}{\pi}\int_0^{\epsilon} \cos(kx)\hat{p}(k)^N \, dk \\
&= \frac{1}{\pi}\int_0^{\epsilon} \cos(kx) e^{Nf(k)} e^{Ng(k)} \, dk \\
&\sim \frac{1}{\pi}\int_0^{\epsilon} \cos(kx) \exp\left(-N\sigma^2 k^2/2 + N\sum_{n=2}^{\infty} \frac{(-1)^n c_{2n}}{(2n)!} k^{2n}\right) \exp\left(N c_\alpha k^\alpha\right) \, dk \ .
\end{aligned}$$

Here we have neglected higher terms in the asymptotic expansion of $g(k)$ since such terms contribute sub-dominantly to the integral. This is intuitively clear since ϵ is small, but it does require some justification (see [1] for a rigorous discussion). We will have to wait until later to see how these neglected terms affect our freedom to choose x.

Since ϵ is small and $\alpha > 2$, the dominant term in the exponents of the above integral is

$-N\sigma^2k^2/2$. We therefore Taylor-expand the other exponentials:

$$
\begin{aligned}
P_N(x) \sim \frac{1}{\pi} \int_0^{\epsilon} & \cos(kx) \exp\left(-N\sigma^2k^2/2\right) \\
& \left[1 + (\text{higher } k^{2n} \text{ terms})\right] \\
& [1 + Nc_\alpha k^\alpha + (\text{higher } k^{m\alpha} \text{ terms})]\; dk \\
\sim & \frac{1}{\pi} \int_0^{\epsilon} \cos(kx) \exp(-N\sigma^2k^2/2) \left[1 + (\text{higher } k^{2n} \text{ terms})\right] dk \\
& + \frac{1}{\pi} \int_0^{\epsilon} \cos(kx) \exp(-N\sigma^2k^2/2) \left[Nc_\alpha k^\alpha\right] dk \\
& + \frac{1}{\pi} \int_0^{\epsilon} \cos(kx) \exp(-N\sigma^2k^2/2) \left[(\text{higher } k^{m\alpha+2n} \text{ terms})\right] dk
\end{aligned}
$$

Again, we neglect terms involving higher powers of k. Therefore

$$
\begin{aligned}
P_N(x) \sim & \frac{1}{\pi} \int_0^{\epsilon} \cos(kx) \exp(-N\sigma^2k^2/2) \left[1 + (\text{higher } k^{2n} \text{ terms})\right] dk \\
& + \frac{1}{\pi} \int_0^{\epsilon} \cos(kx) \exp(-N\sigma^2k^2/2) \left[Nc_\alpha k^\alpha\right] dk\ .
\end{aligned} \tag{13}
$$

Finally, we must re-enlarge the interval of integration. Each term in the integrands above is of the form $k^\beta e^{-Nk^2}$. Though such terms obviously decrease exponentially in N, they are actually equal to 0 at $k = 0$ (except when $\beta = 0$). However, for any fixed ϵ, N may be increased sufficiently for the decreasing exponential factor to dominate the increasing polynomial factor for $k > \epsilon$. Therefore we may enlarge the upper limit of integration to ∞, incurring only an exponentially small error in N as N increases. Moreover, this is true *regardless of the magnitude of* x. This yields:

$$
\begin{aligned}
P_N(x) \sim & \frac{1}{\pi} \int_0^{\infty} \cos(kx) \exp(-N\sigma^2k^2/2) \left[1 + (\text{higher } k^{2n} \text{ terms})\right] dk \\
& + \frac{1}{\pi} \int_0^{\infty} \cos(kx) \exp(-N\sigma^2k^2/2) \left[Nc_\alpha k^\alpha\right] dk\ .
\end{aligned} \tag{14}
$$

Now that we have the large-N asymptotics, we must determine for which values of x the above is valid. This means we must decide how large x may become before the terms we neglected in the above approximations become large relative to the terms we kept.

The errors made during the first and third approximations were exponentially small in N. We will see in Appendix B that the second integral in equation 14 behaves asymptotically like $N/|x|^{1+\alpha}$ as $|x| \to \infty$. Thus even if x grows like a power of N, this term still decreases like a power of N, i.e. much more slowly than exponential decay.

The errors made during the second approximation involved discarding integrals of the above form with powers of k higher than α. Such integrals are still of the form analyzed in Appendix B; asymptotically in x, they behave like $\exp(-x^2/2\sigma^2N)H_{2n}(x/\sigma\sqrt{N})$ or like

$N/|x|^\beta$ for some $\beta > 1 + \alpha$. In either case, they are clearly dominated by the asymptotic behavior (discussed above) for the second integral in equation 14.

Therefore, all discarded terms are small relative to the kept terms in the above approximations as long as we allow x to grow at most like a polynomial in N.

B $F_\beta(z)$

In this appendix, we compute

$$F_\beta(z) \equiv \frac{1}{\pi} \int_0^\infty \cos(wz) e^{-w^2/2} w^\beta \, dw$$

in closed form for all non-negative values of β, and we describe its asymptotic behavior.

B.1 Even-Integral α

Let $\beta = 2n$ for some non-negative integer n. Then:

$$\begin{aligned} F_{2n}(z) &= \frac{1}{\pi} \int_0^\infty \cos(wz) e^{-w^2/2} w^{2n} \, dw \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos(wz) e^{-w^2/2} w^{2n} \, dw \\ &\quad \text{(since the integrand is an even function of } w\text{)} \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iwz} e^{-w^2/2} w^{2n} \, dw \\ &= \text{(since } \sin(wz) e^{-w^2/2} w^{2n} \text{ is an odd function of } w\text{).} \end{aligned}$$

But this is an integral we've seen before, and can be expressed in terms of the Hermite polynomials $H_{2n}(z)$:

$$\boxed{F_{2n}(z) = \frac{(-1)^n}{\sqrt{2\pi}} H_{2n}(z) e^{-z^2/2} \text{ , } n \text{ a non-negative integer}} \tag{15}$$

where we are defining $H_0(z) = 1$. Clearly

$$\boxed{F_{2n}(z) \sim \frac{(-1)^n}{\sqrt{2\pi}} z^{2n} e^{-z^2/2} \text{ , } |z| \to \infty \text{ , } n \text{ a non-negative integer.}} \tag{16}$$

B.2 Odd Integral β

B.2.1 Dawson's Integral

Dawson's integral is defined by

$$D(x) \equiv e^{-x^2} \int_0^x e^{t^2} \, dt \ . \tag{17}$$

Clearly $f(x) \equiv D(x)e^{x^2}$ satisfies the differential equation

$$f'(x) = e^{x^2} \ . \tag{18}$$

From a local analysis of this differential equation (see [1] for generalities, though they don't discuss this example), we can compute an asymptotic expansion for f and thus for D:

$$D(x) \sim \frac{1}{2x} + \frac{1}{4x^3} + \cdots \ , \quad x \to \infty \ . \tag{19}$$

B.2.2 $F_\beta(z)$ for β an odd integer

Let $\beta = 2n+1$ for some non-negative integer n. Then:

$$\begin{aligned}
F_{2n+1}(z) &= \frac{1}{\pi}\int_0^\infty \cos(wz)e^{-w^2/2}w^{2n+1}\,dw \\
&= \frac{(-1)^n}{\pi}\frac{d^{2n+1}}{dz^{2n+1}}\int_0^\infty \sin(wz)e^{-w^2/2}\,dw \\
&\quad \text{(by differentiating under the integral sign)} \\
&= \frac{(-1)^n}{\pi}\frac{d^{2n+1}}{dz^{2n+1}}\mathrm{Im}\left\{\int_0^\infty e^{iwz}e^{-w^2/2}\,dw\right\} \\
&\quad \text{(by Euler's formula)} \\
&= \frac{(-1)^n}{\pi}\frac{d^{2n+1}}{dz^{2n+1}}\mathrm{Im}\left\{e^{-z^2/2}\int_{-iz}^{\infty-iz} e^{-q^2/2}\,dq\right\} \\
&\quad \text{(by change of variables: } q \equiv w - iz)
\end{aligned}$$

The last integral above can be computed by using complex analysis. We consider a rectangular contour in the complex plane going from 0 to R to $R - iz$ to $-iz$ and back to 0 in the limit as $R \to \infty$. The integral along the real axis is

$$\int_0^\infty e^{-q^2/2}\,dq = \sqrt{\frac{\pi}{2}} \ .$$

The integral along the short segment from R to $R - iz$ goes to 0 like $e^{-R^2/2}$. By the change of variables $t \equiv q/i\sqrt{2}$, the integral along the short segment from $-iz$ to 0 is

$$\begin{aligned}
\int_{-iz}^0 e^{-q^2/2}\,dq &= i\sqrt{2}\int_{-z/\sqrt{2}}^0 e^{t^2}\,dt \\
&= i\sqrt{2}e^{z^2/2}D(z/\sqrt{2})
\end{aligned}$$

where D is the Dawson integral discussed above. Since the integrand has no singularities, the integral around the closed contour is 0. Therefore

$$\int_{-iz}^{\infty-iz} e^{-q^2/2}\,dq = \sqrt{\frac{\pi}{2}} + i\sqrt{2}e^{z^2/2}D(z/\sqrt{2}) \ .$$

Substituting this result into the above asymptotic expression yields

$$\boxed{F_{2n+1}(z) = \frac{(-1)^n\sqrt{2}}{\pi}\frac{d^{2n+1}}{dz^{2n+1}}D(z/\sqrt{2})\ ,\ n \text{ a non-negative integer.}} \tag{20}$$

Using the asymptotic expression 19 for D we conclude that

$$\boxed{F_{2n+1}(z) \sim -\frac{(-1)^n}{\pi}(2n+1)!z^{-(2n+2)}\ , |z| \to \infty\ ,\ n \text{ a non-negative integer.}} \tag{21}$$

B.3 Non-Integral β

B.3.1 The Parabolic Cylinder Function

The classical *parabolic cylinder function* may be defined for parameters ν with $\mathrm{Re}(\nu) > -1$ by

$$D_\nu(z) \equiv \sqrt{\frac{2}{\pi}}e^{z^2/4}\int_0^\infty e^{-t^2/2}t^\nu \cos(zt - \frac{\nu\pi}{2})\,dt\ . \tag{22}$$

Using integration by parts, it can be shown that this function satisfies the second-order differential equation

$$D_\nu''(x) + \left(\frac{1}{2} + \nu - \frac{x^2}{4}\right)D_\nu(x) = 0\ . \tag{23}$$

We can compute the asymptotic behavior of D_ν via local analysis of the differential equation:

$$D_\nu(x) \sim x^\nu e^{-x^2/4}\ , \quad x \to +\infty\ . \tag{24}$$

See [1] for details. Note that a more customary presentation defines the parabolic cylinder function as the unique solution of the differential equation 23 which exhibits the asymptotic behavior given by 24.

When ν is an integer, it can be seen directly from the integral representation 22 that $D_\nu(x)$ is an even function (for even ν) or an odd function (for odd ν) of x. For non-integral ν, it can be shown (again using local methods; see [1]) that

$$D_\nu(x) \sim \frac{\sqrt{2\pi}}{\Gamma(-\nu)}e^{x^2/4}|x|^{-\nu-1}\ , \quad x \to -\infty\ . \tag{25}$$

It should be clear why we restricted our attention to non-integral ν: the Γ function has poles at the negative integers. Also, note that this implies that changing the sign of x in the integral 22 dramatically changes the asymptotics.

It is an unfortunate coincidence that Dawson's integral and the parabolic cylinder function are both denoted by D. However, only the cylinder function needs a subscript.

B.3.2 $F_\beta(z)$ for any non-integral β

By using the angle-addition formula for cosines, we see that equation 22 may be written

$$D_\beta(z) = \sqrt{\frac{2}{\pi}} \cos\left(\frac{\beta\pi}{2}\right) e^{z^2/4} \int_0^\infty e^{-t^2/2} t^\beta \cos(zt)\, dt \\ + \sqrt{\frac{2}{\pi}} \sin\left(\frac{\beta\pi}{2}\right) e^{z^2/4} \int_0^\infty e^{-t^2/2} t^\beta \sin(zt)\, dt \ . \quad (26)$$

The first term is an even function of z while the second term is an odd function of z, so

$$D_\beta(z) + D_\beta(-z) = \sqrt{\frac{8}{\pi}} \cos\left(\frac{\beta\pi}{2}\right) e^{z^2/4} \int_0^\infty e^{-t^2/2} t^\beta \cos(zt)\, dt \ .$$

Then it's just a matter of algebra to see that

$$\boxed{F_\beta(z) = \sqrt{\frac{1}{8\pi}} \sec\left(\frac{\beta\pi}{2}\right) e^{-z^2/4} \left[D_\beta(z) + D_\beta(-z)\right] \ , \beta \text{ a positive non-integer.}} \quad (27)$$

It is now clear why this analysis is restricted to non-integral β: when β is an odd integer, $\cos(\beta\pi/2) = 0$.

Now we may apply the asymptotic relations 24 and 25. Clearly the $-\infty$ behavior dominates the $+\infty$ behavior, so

$$F_\beta(z) \sim \frac{1}{2\Gamma(-\beta)} \sec\left(\frac{\beta\pi}{2}\right) |z|^{-\beta-1} \ , \quad |z| \to \infty \ .$$

Using Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

and the trigonometric identity $\sin(x+\pi) = -\sin(x)$ we get

$$F_\beta(z) \sim -\frac{\Gamma(\beta+1)}{2\pi} \sin(\beta\pi) \sec\left(\frac{\beta\pi}{2}\right) |z|^{-\beta-1} \ , \quad |z| \to \infty \ .$$

Finally, applying the double-angle formula for sin gives us

$$\boxed{F_\beta(z) \sim -\frac{\sin(\beta\pi/2)}{\pi} \Gamma(\beta+1) |z|^{-\beta-1} \ , \quad |z| \to \infty \ , \beta \text{ a positive non-integer.}} \quad (28)$$

Though their derivations were quite different, the final asymptotic expressions for $F_\beta(z)$ when β is an odd integer and when β is a non-integer are of the same form. Indeed, equation 28 simplifies to equation 21 if we substitute in an odd integer for β. We may therefore consider equation 28 to be the asymptotic expression for $F_\beta(z)$ for all non-even-integral β (it is clear that, because of the term $\sin(\beta\pi/2)$, equation 28 can't possibly apply when β is an even integer):

$$\boxed{F_\beta(z) \sim -\frac{\sin(\beta\pi/2)}{\pi} \Gamma(\beta+1) |z|^{-\beta-1} \ , \quad |z| \to \infty \ , \beta > 0, \beta \neq 2n \text{ for any integer } n.} \quad (29)$$

References

[1] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. Springer, 1999.

[2] Eugene Lukacs. *Characteristic Functions*. Griffin, second edition, 1970.