18.354 – Nonlinear Dynamics II: Continuum systems
Problem Set 4: Continuum Equations
Due: Tuesday, April 14 (by 1pm in class)

Problem 1: Hydrodynamic conservation laws.

(a) Derive the momentum equation for a fluid using a volume element argument (as opposed to the fluid element argument used in class).

(b) The Reynolds Transport Theorem (RTT) states that for a scalar field $f(\mathbf{x}, t)$,

$$\frac{d}{dt}\int_{V(t)} f(\mathbf{x}, t)dV = \int_{V(t)}\left[\frac{\partial f}{\partial t} + \nabla.(\mathbf{u}f)\right]dV$$

where $V(t)$ indicates a material control volume, i.e., a volume element that moves with the local fluid velocity $\mathbf{u}(\mathbf{x}, t)$. If the density ρ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\mathbf{u}) = 0,$$

prove the special form of the RTT for another scaler field $g(\mathbf{x}, t)$:

$$\frac{d}{dt}\int_{V(t)} \rho g(\mathbf{x}, t)dV = \int_{V(t)} \rho\frac{Dg}{Dt}dV,$$

where D/Dt represents the material derivative $D/Dt = \partial/\partial t + \mathbf{u}.\nabla$.

Problem 2: Reynolds numbers.

The kinematic viscosity ($\nu = \mu/\rho$) of air is roughly $10^{-5}\text{m}^2\text{s}^{-1}$ and for water it is $10^{-6}\text{m}^2\text{s}^{-1}$. Find the kinematic viscosities of some other fluids (e. g. Honey, Liquid Helium) and estimate the Reynolds number for a least three different physical flow configurations (e.g. airplane flight, stirring of a coffee cup, swimming of bacteria, etc).

Problem 3: The stress tensor.

(a) Calculate the velocity gradient tensor $\nabla\mathbf{u}$, the rate of strain tensor $\mathbf{E}$ and the vorticity tensor $\mathbf{\Omega}$ for the shear flow $\mathbf{u} = (Ay, 0, 0)$. Sketch the velocity field.

(b) Fluid motion in the neighbourhood of a point can be decomposed into a combination of translational flow, rotational flow and straining flow. For the shear flow in (a) we can write $\mathbf{u} = \mathbf{u}_s + \mathbf{u}_r = \frac{1}{2}(Ay, Ax, 0) + \frac{1}{2}(Ay, -Ax, 0)$. Calculate $\nabla\mathbf{u}$, $\mathbf{E}$ and $\mathbf{\Omega}$ for $\mathbf{u}_s$ and $\mathbf{u}_r$, and sketch the velocity fields.

Problem 4: SOLUTIONS OF THE NAVIER-STOKES EQUATIONS.

The Navier-Stokes equations can be solved analytically for the following flow geometries. In each problem the fluid has viscosity μ and density ρ. Having obtained a solution, sketch the corresponding velocity profile. The solution in (d) is time dependent, so plot the velocity profile at two or three different times. Briefly comment on your solutions (i. e. what would be the effect of changing viscosity? or density? or changing the frequency in (d)?).

(a) Flow of fluid between two rigid boundaries located at $y = \pm h$ under a constant pressure gradient $P = -dp/dx$.

(b) The same as (a) but for flow inside and along the axis of a circular cylinder of radius h. (*use cylindrical coordinates*)

(c) Flow between a cylinder of radius R_1 inside another cylinder of radius R_2. The inner cylinder rotates at angular velocity ω_1 and the outer cylinder rotates at angular velocity ω_2. (*use cylindrical coordinates*)

(d) Fluid lying in the plane $0 < y < \infty$, above an oscillating flat plate moving to and fro in the x-direction with velocity $U \cos \omega t$. Seek a solution for the velocity field of the form
$$\mathbf{u} = (u(y,t), 0, 0) = (\mathrm{Re}[f(y)e^{i\omega t}], 0, 0).$$

The first two flows are called Poiseuille flows, after the physician who first studied them in connection with blood flow. Their instability at high Reynolds number constitutes one of the most important problems in fluid dynamics. The third flow is cylindrical Couette flow.

Problem 5: COFFEE CUP EXPERIMENT EQUATIONS.

(a) Take a cup of water. With a spoon, stir the fluid until the flow is roughly in solid body rotation. Estimate the initial rotational angular velocity of the fluid, and time how long it takes for the fluid to spin down to rest. How does this compare with our calculations in class ? How does the spin down time change if you use a more viscous fluid, like honey, or if you change the size of your container? Write a brief description of your experimental method and results.

(b) Our solution to the spin-down of a coffee cup was composed of Bessel functions. Use MATLAB to make some plots of the velocity profile for flow in your coffee cup at various stages of the spin down, and describe any difficulties you encounter. Provide a printout of your code for calculating the velocity profile.

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