18.354 – Nonlinear Dynamics II: Continuum systems
Problem Set 2: DIFFUSION
Due: Thursday March 12 (by 1pm in class)

Problem 0: PROPOSAL OF COURSE PROJECT

Write a short proposal of your final project. This proposal should be one page long, include a couple of references and describe what you would like to do. While choosing a topic and implementing it, you should show application of concepts connected to the class material. A relevant experiment, a numerical implementation or a theoretical review are all valid approaches. Original work will carry extra credit. You may want to browse through the course schedule to see the topics that we will be covering before and after spring break (elasticity, hydrodynamics, rotating flows, active matter). I will be available throughout the week after Spring Break to further discuss projects with you.

Problem 1: GREEN'S FUNCTIONS

(a) Use the Fourier method to solve for the Green's function of the diffusion equation.

(b) Find the Greens functions for the continuum equations you derived in Problem 4 of Set 01. *Hint: to account for the drift in the second case move into the frame of reference of the translating cloud by considering a solution $f(x - ct, t) = g(x, t)$.*

Problem 2: RANDOM WALK SIMULATION

In class we discussed the random walk. The point of this problem is to use MATLAB[i] to verify our assertions that a cloud of random walkers is well described by the diffusion equation.

(a) Write a simple MATLAB program to simulate the motion of random walkers. Start with all the walkers at a fixed point and show that if they are allowed to take random steps to either the left or the right, they disperse according to the $\sqrt{t}$ law. (*Hint: If you prefer, an alternative to letting the random walkers take discrete steps to the left or right of size $\delta = 1$ is to use the function* **randn** *to generate random data with zero mean and unit variance*). Provide a printout of your MATLAB code.

(b) Show that the distribution of walkers after many time steps of the cloud is well approximated by the analytic solution to the diffusion equation.

Problem 3: RETURN PROBABILITIES

The aim of this problem is to recover a famous result: a random walker starting at the origin will (with probability 1) return to the origin in 1 and 2 dimensions, but not in 3! (ie. there is only some probability less than 1 that the walker will return in 3D). First, we wish to understand one random walker. To do this, let $p(x,t)$ denote the probability distribution for a walker (Note: the concentration $n(x,t)$ is related to $p(x,t)$ by the total number of particles N in the system: $n(x,t) = Np(x,t)$). Therefore, the probability of finding the walker at time t, between x and $x + dx$ is $\int_x^{x+dx} p(x,t)dx \approx p(x,t)dx$. In addition, since $n(x,t)$ satisfies the diffusion equation, so does $p(x,t)$.

Part A Consider a random walker meandering around in an infinite domain. Take the walkers starting location at the origin, so that $p(x,0) = \delta(x)$. Since, $p(x,t)$ solves the diffusion equation, $\frac{\partial}{\partial t}p(x,t) = \nabla^2 p(x,t)$, the solution for $p(x,t)$ is exactly the Greens function. An extension of the calculation in class to any dimension, modifies the Greens function to:

$$p(x,t) = \frac{1}{(4\pi t)^{d/2}}e^{-r^2/4t} \tag{1}$$

Here d is the dimension of the domain, while r is the distance from the origin (the radial polar coordinate).

(a) For each dimension, $d = 1, 2, 3$: if you observed the walker from time $t = 1$ to time $t = T$, how much time would you expect to see the walker near the origin? (*Hint: If V is the volume of a small ball around the origin, then $Vp(0,t)$ is roughly the probability that the walker is contained inside V at time t. If τ is the time spent inside V, then the expected value of τ is $\langle \tau \rangle = V \int_1^T p(0,t)dt$, ie. think of the particle having probability $p(0,t)$ of spending time dt near the origin.*)

(b) After observing for time T, for which dimension would you expect to see the walker near the origin the most (and which dimension the least)?

(c) What happens to $\langle \tau \rangle$ for $d = 1, 2, 3$ when $T \to \infty$? Convince yourself that this implies the particle should return to the origin infinitely often in 1 and 2 dimensions, while in $3D$, the random walker will only spend a finite amount of time near the origin (ie. if you get lucky may return once or twice, but after a time will eventually be lost).

Part B The calculation in part A explicitly requires an infinite domain. The result no longer holds if we restrict the walker to live inside a 3D container (regardless of the shape!). Consider a 3D domain with volume Ω. Again let $p(x,0) = \delta(x)$. After long times, solutions to the diffusion equation approach a constant. The only appropriate constant is $p(x,t) \to 1/\Omega$ (ie. a uniform probability distribution). Argue that the walker spends an expected time $\langle \tau \rangle \to \infty$ at the origin, as $T \to \infty$. After watching a random walker for a very long time, which would you expect to see spend more time at the origin: a walker on a 1D infinite line OR a walker contained inside a 3D box?

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