18.336 Numerical Methods of Applied Mathematics -- II
Spring 2009

MIT Dept. of Mathematics
Benjamin Seibold

18.336 spring 2009
Problem Set 1

Out Thu 02/12/09 Due Thu 02/26/09

Problem 1
The telegraph equation $u_{tt} + 2du_t = u_{xx}$ describes the evolution of a signal in an electrical transmission line (`en.wikipedia.org/wiki/Telegraph_equation`). Consider $x \in [-\pi, \pi)$ with periodic boundary conditions. Find the solution by a Fourier approach. Show that the waves e^{ikx} travel with frequency dependent velocities, while being damped with time.

Problem 2
Consider the 1d Poisson equation

$$\begin{cases} -u_{xx} = f & \text{in }]0,1[\\ u = 0 & \text{on } \{0,1\} \end{cases} \tag{1}$$

with $f(x) = \sin(\phi(x))(\phi_x(x))^2 - \cos(\phi(x))\phi_{xx}(x)$, where $\phi(x) = 9\pi x^2$. Run the program `mit18336_poisson1d_error.m` from the course web site, which approximates (1) by a linear system, based on the approximation $u_{xx} \approx D^2u$, where $D^2u(x) = \frac{1}{h^2}(u(x+h) - 2u(x) + u(x-h))$.

1. Use Taylor expansion to explain the observed error convergence rate.
2. Modify the system matrix, such that fourth order error convergence is achieved. Show error convergence plots.
3. Return to the original system matrix based on $u_{xx} \approx D^2u$. Now change the right hand side vector from $f_i = f(ih)$ to $f_i = f(ih) + \frac{h^2}{12}D^2f(ih)$. Prove that this modification yields fourth order accuracy, and produce an error convergence plot that verifies this result.[1] How does the error constant compare to fourth order system matrix in part 2.?
4. Change the right hand side to

$$f(x) = \begin{cases} 1 & \text{for } x \leq \frac{1}{2} \\ 0 & \text{for } x > \frac{1}{2} \end{cases}$$

and report and explain the new error convergence rate for the various solution approaches.

[1] This trick is called *deferred correction*.

Problem 3

Write a program that approximates the *biharmonic equation*

$$\begin{cases} u_{xxxx} = f & \text{in }]0,1[\\ u = 0 & \text{on } \{0,1\} \\ u_x = 0 & \text{on } \{0,1\} \end{cases}$$

with $f = 24$. Investigate the accuracy of the numerical approximations you obtain.