14.3 Lanczos Method

$$\text{Lanczos} = \text{Arnoldi for } (A^T = A) \tag{14.19}$$

H is then tridiagonal.

14.4 Generalized Minimum Residuals (GMRES)

Find $x \in \mathcal{K}_k : \|Ax - b\|_2$ is minimized. $x \in \mathcal{K}_k \Rightarrow x = Q_k y \Rightarrow \|AQ_k y - b\|$.

We have

$$AQ_k = Q_{k+1}\tilde{H} \tag{14.20}$$

$$\tilde{H} = \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ h_{21} & \ddots & \vdots \\ & \ddots & h_{nn} \\ & & h_{n+1,n} \end{bmatrix} \tag{14.21}$$

$$\begin{aligned} \left\| Q_{k+1}\tilde{H}y - b \right\|_2 &= \left\| \tilde{H}y - Q_{k+1}^* b \right\|_2 \\ &= \left\| \tilde{H}y - \|b\| e_1 \right\|_2 \end{aligned} \tag{14.22}$$

Minimizing $\left\| \tilde{H}y - \|b\| e_1 \right\|_2$ is a standard least square problem solved easily using Givens rotations ($\tilde{H}$ is Hessenberg, so the Givens rotations have only one subdiagonal to kill).